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Chaotic Diffusion of Localized Turbulent Defect and Pattern Selection in Spatiotemporal Chaos.

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Abstract. – Coupled-map lattices are investigated as a model for the spatiotemporal chaos. Patterns with some wavelengths are selected through a chaotic motion of domain boundaries. Localized defect which separates two domains with antiphase is found. It changes chaotically in time and moves randomly in space. The diffusion coefficient and Kolmogorov-Sinai entropy of a defect are calculated. A novel phase transition at the collapse of a pattern is studied in connection with a crisis in a high-dimensional dynamical system.

Characterization of spatiotemporal chaos is one of the most important problems in the nonlinear science of our age. It is essential not only for the understanding of the nature of turbulence in fluids, optics, solid-state physics and chemistry, but also for a study of coupled dynamical systems in biology and technology [1]. As a simple model for a study of the spatiotemporal chaos, a coupled-map lattice (CML) has been proposed and simulated [2-5].

A CML is a dynamical system with a discrete time and space, and a continuous state [2, 3, 6-9]. Though there are various possible models, we restrict ourselves to the following diffusive coupling model here [2, 3, 6, 10]:

$$x_{n+1}(i) = (1-\varepsilon) f(x_n(i)) + \varepsilon/2[f(x_n(i+1)) + f(x_n(i-1))], \tag{1}$$

where n is a discrete time step and i is a lattice point ($i = 1, 2, ..., N =$ system size). Throughout the letter a periodic boundary condition is adopted. The mapping function $f(x)$ is chosen to be the logistic

$$f(x) = 1 - ax^2,$$

but the phenomena to be shown later can be seen in a wide class of mappings and in other types of couplings.

As has been reported, the model shows a rich variety of phenomena, such as period-doubling bifurcations of kinks, spatial bifurcation, spatiotemporal intermittency, and pattern competition intermittency [2, 3, 8, 11]. In the present letter, we investigate the simplest case of pattern selection, that is, a selection of a mode with wavelength 2. Novel discoveries are the existence of localized chaos as a defect which separates two domains of different phases and the diffusion of such defects. Although our system is deterministic, defects exhibit a Brownian motion triggered by the chaos. This is the first observation that chaos induces the diffusion in *real space*, not in phase space. A new type of phase transition in pattern dynamics is also reported.

Let us survey the phases of patterns in the model (1). As the nonlinearity a is increased, period-doubling of kinks to chaos is observed [3]. For weak nonlinearity ($a < 1.55$), positions of domains separated by kinks do not move and a chaotic motion is confined within each domain [6]. Hence domains of various sizes coexist. The distribution of domain sizes depends on initial conditions. As the nonlinearity is increased further, domain boundaries start to move in the space. Domains of larger sizes are no longer stable. Only some sizes of domains are selected to produce a pattern with some wavelength k_p. If the coupling ε is small ($\varepsilon < 0.15$), a domain with $k_p = 1/2$ (zigzag structure) is selected. For larger couplings, sizes of domains are larger (*e.g.*, $k_p = 1/6$ or $1/8$ for $\varepsilon = 0.3$). Domains are selected so that the temporal motion within them is simpler, that is periodic with a short period. For example, the zigzag pattern for $1.63 < a < 1.74$ and $\varepsilon = 0.1$ is periodic with period 2.

Chaos is suppressed through this selection of pattern, which is confirmed by the decrease of the Kolmogorov-Sinai (KS) entropy of our lattice system (1).

Let us focus on the zigzag pattern here (see fig. 1 for space-amplitude plots). The zigzag pattern is easily characterized by the condition

$$(x_n(i+1) - x_n(i))(x_n(i) - x_n(i-1)) < 0\,, \tag{2}$$

or the condition $(x_n(i+1) - x^*)(x_n(i) - x^*) < 0$ with $x^* = (\sqrt{1+4a} - 1)/(2a)$, an unstable fixed point of the logistic map x. (Both the conditions give the same results in our case.) Regions of two different phases (which correspond to the sign of the two terms in (2)) are separated by a defect (see fig. 1).

As the nonlinearity a is increased, the following change in pattern dynamics proceeds.

i) Frozen pattern: a zigzag structure is fixed in space. Temporal change of domain boundary is chaotic (has a positive Lyapunov exponent), but it does not move in space. The temporal change of zigzag region is period-2 if there is no defect. The motion is modulated a little by a defect.

ii) Diffusion of defect: the zigzag pattern is modulated quasi-periodically in time. The chaos in a domain boundary is still localized (see fig. 1), but it can move in space. The motion of a defect is represented as a random walk as is shown later.

iii) Diffusion of defect in chaotic media: the zigzag pattern is modulated chaotically in time, but the pattern itself is fixed in space, that is, eq. (2) is satisfied in a single-zigzag region. A defect as a domain boundary between two regions with antiphase is still localized, changes chaotically in time, and moves in space (see fig. 2).

In the regimes ii) and iii), defects are not created spontaneously and they pair-annihilate by collisions. If the motion of defect is a random walk, the number is expected to decrease with $(n)^{-1/2}$, with the time step n. Numerical simulations from arbitrary chosen initial conditions exhibit such decay. In the regime ii), initial decay is much slower, but approaches the form of $(n)^{-1/2}$ asymptotically.

(1) Throughout the paper, Lyapunov exponents are calculated from products of Jacobi matrices of the global map acting on the entire lattice [6].

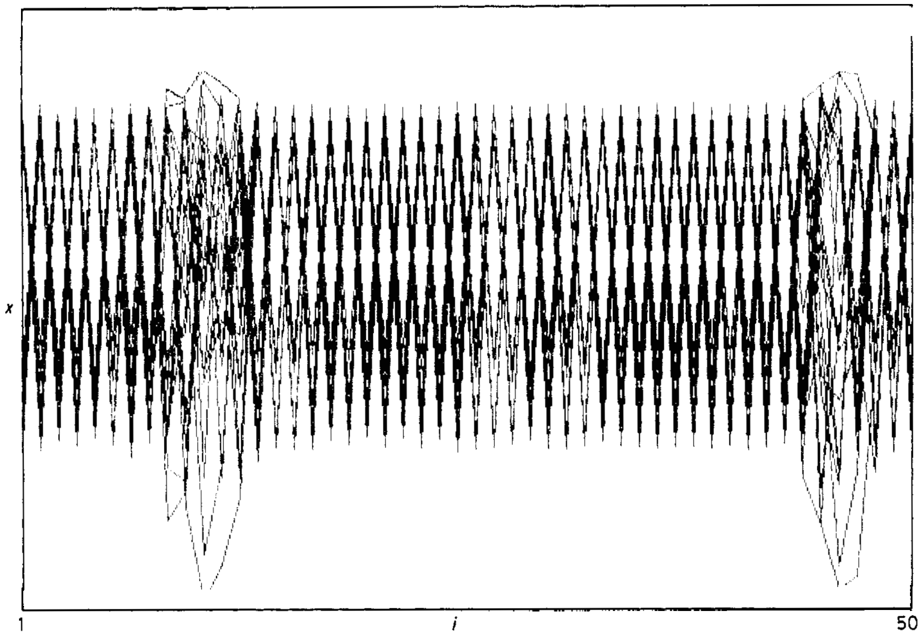

Fig. 1. – Space-amplitude plot for the coupled logistic lattice (1). Amplitudes $x_n(i)$'s are overlayed for 20 time steps after 1000 iterations of transients. System size $N = 50$, $a = 1.81$, $\varepsilon = 0.1$: the zigzag region is temporally quasi-periodic, while the defects move around in space, till they disappear by collisions.

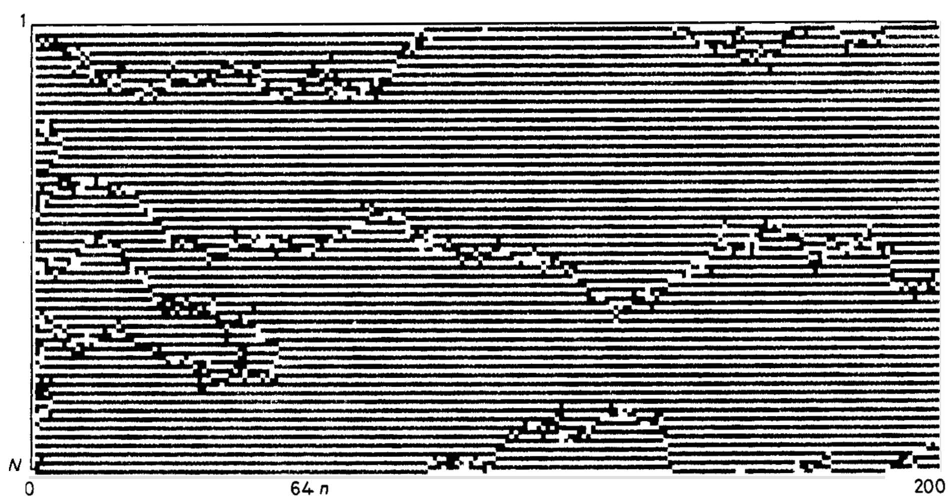

Fig. 2. – Space-time diagram for the coupled logistic lattice (1), with $a = 1.85$, $\varepsilon = 0.1$, $N = 100$ and starting with a random initial condition. Every 64th time step is plotted from 0 to $200 \cdot 64$. If $x_{64n}(i)$ is larger than x^*, the corresponding space-time pixel is painted as black, while it is left blank otherwise.

iv) Defect turbulence and fully developed turbulence: a zigzag region is no longer stable and a spontaneous pair-creation of defects is seen, even if we start from a complete zigzag pattern. If a is smaller ($a < 1.91$ for $\varepsilon = 0.1$), a zigzag pattern is still dominant and its motion has a long time tail [11]. If a is larger, the dynamics of pattern is well approximated by the Markovian dynamics.

For $\varepsilon = 0.1$, the phase change occurs at $a = 1.74$ (i)$\rightarrow$ii)), $a = 1.83$ (ii)$\rightarrow$iii)), and $a = a_c = 1.88$ (iii)$\rightarrow$iv)), respectively. These phases can be distinguished by the following two methods.

Spatial return maps. – Two-dimensional plots $(x_n(i), x_n(i+1))$ are used to characterize a short-ranged structure [3, 8]. If there is no defect, they give two separated closed curves in region ii) and two separated surfaces in iii), as has been seen in two-dimensional mappings [2]. In the presence of defects, some scattered points connect the two separated regions (see fig. 3). These scattered points correspond to the motion of a defect.

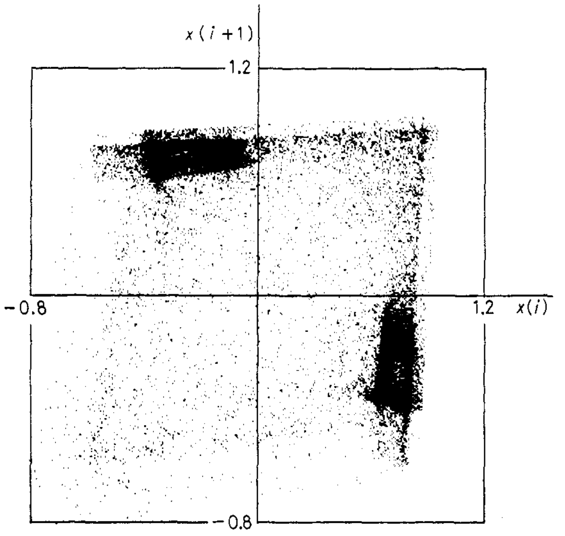

Fig. 3. – Spatial return map for the model (1) with $a = 1.83$, $\varepsilon = 0.1$, $N = 100$ and a random initial condition. $(x_n(i), x_n(i+1))$ are plotted for the entire lattice for $500 < n < 1000$. Here some defects exist.

Lyapunov spectra. – In the regime ii), first few Lyapunov exponents are close to zero in the absence of defects, while some are positive in the presence of defects. The number of positive exponents is proportional to the number of defects. This clearly shows that a temporal motion of a defect is chaotic. In the regime iii), there are positive Lyapunov exponents even in the absence of defects. In the presence of defects, however, some exponents are much larger than the above positive values. The number of such exponents is again proportional to the number of defects. Thus the chaos associated with a defect is of a different nature from the chaos in the motion of a zigzag pattern.

In the regime iv), Lyapunov spectra change smoothly with the index of exponent, which is typical in the fully developed spatiotemporal chaos [6].

Let us discuss the motion of a defect in a little more detail. Here we take a viewpoint that the dynamics of our system is represented as inherent motions of a defect and of a zigzag region and an interaction between them. This picture is valid as a first step towards the understanding of dynamics.

First we estimate the KS entropy of a defect, by choosing a single-defect initial state

$$x_n(i) = x^* + c(-1)^n + \mathrm{rnd}(b)\,, \tag{3}$$

where $\mathrm{rnd}(b)$ is a uniformly distributed random number in $(-b, b)$ and the constants c and b are chosen to generate a zigzag pattern. System size N is odd so that a single defect always exists (recall the periodic boundary condition). In the regime ii), the KS entropy of a defect is estimated by the sum of positive Lyapunov exponents, while, in iii), it is estimated by the sum of exponents larger than the maximum Lyapunov exponent in the absence of defects.

This calculation of KS entropy here is based on the above picture of dynamics and on an approximation: a possible effect of a defect on the global motion of a zigzag region is neglected. As a matter of fact, Lyapunov spectra for the zigzag region are slightly modified by the existence of a defect. This modification, however, can be negligible if the density of defect is small.

The KS entropy of a defect is shown in fig. 4. The logarithm of the KS entropy increases linearly with a.

To study the Brownian motion of a defect quantitatively, we choose again a single-defect initial state (3) with $N = 255$. The position of a defect I_n is calculated by the condition (2). Numerical data are well fitted by the expression

$$\langle (I_n - I_0)^2 \rangle = 2Dn\,, \tag{4}$$

from which the diffusion coefficient D is calculated. The bracket $\langle \dots \rangle$ represents the ensemble average for a set of initial conditions (64 samples). An effect of a boundary is taken account of for the calculation of I_n.

The diffusion coefficient is shown in fig. 4 as a function of a. Rapid increase of the diffusion coefficient (by 10^2 times) is prominent.

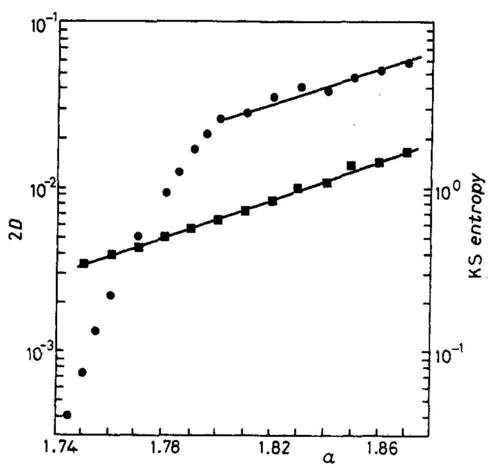

Fig. 4. – Diffusion coefficient D (●) and the KS entropy of a defect (■) as a function of parameter a. See text for the method of calculations.

The diffusion coefficient is shown in fig. 4 as a function of a. Rapid increase of the diffusion coefficient (by 10^2 times) is prominent.

The diffusion process of defect in our system consists of the interaction of chaos of a defect and the motion of zigzag region. Randomness is generated by the former part. If the motion of the zigzag region is period-2 (in the region i)), the interaction has too many constraints to allow for a motion of defect. A defect is pinned at its position. If the motion of a zigzag region is quasi-periodic or chaotic, the interaction allows for a motion of a defect.

We note that in region iii), the increase of the diffusion coefficient is proportional to that of the KS entropy of a defect. This may be explained as follows: the time scale involved in the random force is expected to be the inverse of the KS entropy, since the KS entropy gives the rate of loss of memory of the initial state. The diffusion coefficient is proportional to the inverse of time scale and thus is proportional to the KS entropy of a defect.

This argument holds only if we neglect a possible change of the strength of interaction induced by the change of a motion of a zigzag region. In regime iii), this approximation seems to be valid according to the data of Lyapunov spectra. In regime ii), the interaction term is much more sensitive to the parameter a, which is thought to be a reason for the faster decrease in the diffusion coefficient in the regime.

The diffusion of a kink is observed in a cellular automaton [12], where the randomness comes from an initial random condition. The random walk in our problem arises not from the randomness in the zigzag pattern, but from the chaotic motion of a defect. Even if we take a homogeneous-zigzag initial condition ($b \to 0$ in (2)), the diffusion coefficient D is the same.

Another interesting aspect is a critical behaviour of the collapse of the zigzag pattern. As a set of order parameters for a pattern we choose a distribution of domain size, which is defined as the minimum spatial length which satisfies the condition: $(x_n(i) - x^*)$ has the same sign in the domain.

The distribution function $Q(k)$ for the spatial length k is calculated from a space-time average through the entire lattice and many iterations. After transients of a large number of iterations, $Q(1)$ approaches 1 and $Q(k)$ for $k \neq 1$ goes to zero in regimes ii) and iii), since a single-zigzag domain covers the whole space (for even N). In regime iv), $Q(k)$ for $k \neq 1$ no longer vanishes. The value $1 - Q(1)$ gives a «disorder parameter». The increase of this value with a is roughly fitted by $1 - Q(1) \approx (a - a_c)^\beta$ with $\beta = 1.1(\pm 0.2)$, where a_c (≈ 1.88 for $\varepsilon = 0.1$) is the onset parameter of collapse of a zigzag pattern (border between regimes iii) and iv)). In the fully developed chaos, $Q(k)$ for large k decays exponentially with k. $Q(k)$ is useful as a set of order parameters in the pattern dynamics in spatiotemporal chaos [13].

Spontaneous collapse of a zigzag pattern is explained as a crisis [14] in a high-dimensional dynamical system. Indeed, the model (1) with $N = 2$ (two-coupled logistic map) shows a sudden broadening of attractor due to a crisis from a zigzag pattern at $a' = 1.92$, slightly larger than a_c. The map could be derived from our CML if the zigzag pattern were uniform, *i.e.* $x_n(i) = x_n(i+2)$ for all i. The discrepancy between a_c and a' is due to a spatial modulation of zigzag pattern by chaos. Once the crisis occurs at a lattice site, spatial coupling induces the propagation of crisis to other sites.

For $a \approx a_c$, the collapse of zigzag pattern is hardly observed. At these parameter regimes, the dynamics is governed by a pair-creation of defects by the crisis, their Brownian motion, and collisions of defects, which lead to a pair-annihilation or a complicated pattern. The formation of turbulence by this mechanism is common with the «soliton turbulence» in coupled circle lattices [8]. At these parameter regimes, temporal-power spectra of the spatial Fourier mode $k = 1/2$ show a flickerlike noise at low frequency [11].

The irregular motion of defect has been observed in the experiments of convection in fluids and in liquid crystals with large aspect ratio [15, 16]. In these examples, the irregular motion is associated with a deterministic mechanism, although the relevance of chaos is not yet clarified. Our example gives the possibility that this kind of irregular motion in space may be originated in the temporal chaos.

In a 2-dimensional CML, Brownian motion of the chaotic string is observed, which separates two zigzag regions out of phase.

To sum up, we report a deterministic Brownian motion of a localized chaotic defect. Such localized turbulent defect may explain some localized dynamical disorder in the experiments of nonlinear systems with spatial degrees of freedom.

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