SELF-CONSISTENT PERRON–FROBENIUS OPERATOR FOR SPATIOTEMPORAL CHAOS

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A self-consistent Perron–Frobenius operator is introduced. The invariant measure in a subspace for the spatiotemporal chaos of a coupled map lattice is calculated through the operator. Some applications to spatiotemporal intermittency transitions and pattern dynamics are briefly presented.

To understand spatiotemporal chaos is one of the most important problems in nonlinear dynamics at present. As a simple model for spatiotemporal chaos, coupled map lattices (CML) have been proposed [1–3] and have been extensively investigated [4–12].

Reasons that we use a CML here are: (i) it is numerically efficient, (ii) dynamical system theories of low-dimensional chaos can be extended to apply to spatially extended systems, (iii) statistical mechanical treatment is possible, and (iv) it provides a conceptual basis for the study of phenomena in spatially extended systems.

A CML is a dynamical system with a discrete time, discrete space, and continuous state [1–12]. Although there are various kinds of coupling between nearby lattice points which may be used in a CML, we restrict ourselves here to the following diffusive coupling case here:

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\tfrac{1}{2}\epsilon[f(x_n(i+1))+f(x_n(i-1))]\,, \tag{1}$$

where n is a discrete time step and i is a lattice point ($i=0, 1, ..., N-1$; N is the system size) with a periodic boundary condition. Here the mapping function $f(x)$ is chosen to be the logistic map $f(x)=1-ax^2$ or some other maps.

The Perron–Frobenius (PF) operator has been a powerful aid in the study of the statistical mechanics of low-dimensional chaos [13–15]. The operator has first been extended to spatially extended systems in ref. [16], where the local structure theory of cellular automata is constructed. In the present Letter we combine the above two approaches by introducing a formulation of the self-consistent Perron–Frobenius operator and apply it to the spatiotemporal chaos.

First we start with a measure $\rho(x(1), x(2), ..., x(N))$ on the total lattice (N-dimensional dynamical system). The PF operator for the entire dynamical system is given by [13–15]

$$H^{\mathrm{PF}}\rho(x(1), ..., x(N))=\sum_{y(i)=\mathrm{preimages}}\frac{\rho(y(1), ..., y(N))}{J(y(0), ..., y(N-1))}\,, \tag{2}$$

where the sum is over all possible sets of $(y(i))$, preimages of $x(i)$ (i.e., $y(i)\to x(i)$ by the map (1)) and $J(y(0), ..., y(N-1))$ is the Jacobian of the CML transformation (1).

Here the preimages of our system are calculated as follows [5]:

First, note that our model consists of two successive transformations, i.e., $y(i)\to x'(i)=f(y(i))$ and the spa-

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tial average by $x(i)=(1-\epsilon)x'(i)+\frac{1}{2}\epsilon[x'(i+1)+x'(i-1)]=\sum D_{il}x(l)$. Here the tridiagonal diffusion matrix is given by $D_{ij}=(1-\epsilon)\delta_{i,j}+\frac{1}{2}\epsilon(\delta_{i+1,j}+\delta_{i-1,j}+\delta_{0,N-1}+\delta_{N-1,0})$, where $\delta_{i,j}$ is a Kronecker δ (note the periodic boundary condition).

The inverse process of the latter is given just by the inverse of the tridiagonal matrix D_{ij}, which leads to

$$x'(j)=\sum_{l=0}^{N-1} D_{jl}^{-1}x(l)\equiv\sum_{l=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1}\frac{\exp[2\mathrm{i}k\pi(l-j)/N]}{1-2\epsilon\sin^2(k\pi/N)}x(l)\,. \tag{3}$$

The inverse of the nonlinear transformation is just $y(j)=f^{-1}(x'(j))$, where $f^{-1}(x)$ is the inverse function of $f(x)$ (for the logistic map it is given by $\pm\sqrt{(1-x)/a}$. Thus the preimages of CML (1) are given by

$$y(j)=f^{-1}\left(\sum_l D_{jl}^{-1}x(l)\right). \tag{4}$$

Using the chain rule, we get the following expression for the entire PF operator:

$$H^{\mathrm{PF}}\rho(x(1),x(2),...,x(N))=\frac{1}{\det D}\sum_{y(1),...,y(N)}\frac{\rho(y(1),...,y(N))}{\prod_{j=1}^{N}|f'(y(j))|}\,, \tag{5}$$

where $\sum_{y(1),...,y(N)}$ runs over all possible solutions of (4).

Since this N-dimensional distribution is too difficult to treat directly, we introduce the following projection to the k-dimensional subspace $(x(1), x(2), ..., x(k))$:

$$\rho(x(1),x(2),...,x(k))=\int...\int \mathrm{d}x(0)\,\mathrm{d}x(k+1)\,\mathrm{d}x(k+2)\,...\,\mathrm{d}x(N-1)\,\rho(x(0),x(1),...,x(N-1))\,. \tag{6}$$

Integrating out (5) by $\mathrm{d}x(0)\,\mathrm{d}x(k+1)\,\mathrm{d}x(k+2)\,...\,\mathrm{d}x(N-1)$, neglecting a spatial correlation in ρ longer than k, and after some transformations of variables, we obtain the following expression for the subspace distribution function:

$$H^{\mathrm{SPF}}\rho(x(1),x(2),...,x(k))=\frac{1}{\det D'(k)}\int\int\sum_{y(1),...,y(s)}\mathrm{d}y(0)\,\mathrm{d}y(k+1)$$
$$\times\frac{\rho(y(1),...,y(k))P(y(2),y(3),...,y(k)\,|\,y(k+1))P(y(1),y(2),...,y(k-1)\,|\,y(0))}{\prod_{j=1}^{k}|f'(y(j))|}\,, \tag{7}$$

where the conditional probability P is given by

$$P(y(2),y(3),...,y(k)\,|\,y(k+1))=\rho(y(2),y(3),...,y(k),y(k+1))/\rho(y(2),y(3),...,y(k)) \tag{8}$$

and the preimages $(y(1), y(2), ..., y(k))$ are given by the solution of

$$y(j)=f^{-1}(D_{jl}'^{-1}(k)x_l)-\tfrac{1}{2}\epsilon[f(y(0))\delta_{j,1}+f(y(k+1))\delta_{j,k}]\,. \tag{9}$$

The matrix $D'(k)$ is the k-dimensional diffusion matrix D_{ij} of size k without a periodic boundary (i.e., $(1-\epsilon)\delta_{i,j}+\frac{1}{2}\epsilon(\delta_{i+1,j}+\delta_{i-1,j})$).

The above equation has a simple interpretation. First we write the PF operator for the CML of size k, with the boundary at $x(0)=y(0)$ and $x(k+1)=y(k+1)$. Then we calculate the probability that a spatial sequence of k lattice points takes a set of values $(y(0), y(1), ..., y(k-1))$ and $(y(2), y(3), ..., y(k+1))$ *self-consistently* from our k-dimensional probability distribution function. By integrating out the probability of the set of the values $y(0)$ and $y(k+1)$, with the k-dimensional PF operator, we get the above self-consistent Perron–Frobenius (SPF) operator. Thus our SPF is a PF operator for a k-lattice system with a heat bath at both ends, the strength of which is determined self-consistently.

The projected invariant measure $\rho^*(x(1), ..., x(k))$ onto a k-dimensional space is obtained as the fixed point function of the above operator (7).

In the following examples, we discuss cases of very small subspaces. The simplest case is the one-body approximation given by $k=1$. In this case the SPF is given by

$$H^{\mathrm{SPF}}\rho(x)=\frac{1}{1-\epsilon}\int\int \sum_{y=f^{-1}([x-(\epsilon/2)(y_0+y_2)]/(1-\epsilon))} \frac{\rho(y)\rho(y_0)\rho(y_2)}{|f'(y)|}\,\mathrm{d}y_0\,\mathrm{d}y_2\,. \tag{10}$$

In the fully-developed spatiotemporal chaos [4], the above one-body approximation is fairly accurate. As in fig. 1, the fixed point function of our SPF (10) for the logistic lattice agrees quite well with the distribution function obtained by a direct numerical simulation of (1).

The second simplest case is the two-body approximation, in which the SPF for $\rho(x(1), x(2))$ is given by

$$H^{\mathrm{SPF}}\rho(x_1,x_2)=\frac{1}{(1-\epsilon)^2-(\epsilon/2)^2}\int\int\sum_{y_1,y_2}\frac{\rho(y_1,y_2)\rho(y_0,y_1)\rho(y_2,y_3)}{\rho(y_1)\rho(y_2)|f'(y_1)f'(y_2)|}\,\mathrm{d}y_0\,\mathrm{d}y_3\,, \tag{11}$$

where $\rho(y_2)=\int\rho(y_1, y_2)\,\mathrm{d}y_1$ and the preimages (y_1, y_2) are given by the solutions of

$$f(y_1)=(1-\epsilon)x_1+\tfrac{1}{2}\epsilon[f(x_2)+y_0],\qquad f(y_2)=(1-\epsilon)x_2+\tfrac{1}{2}\epsilon[f(x_1)+y_3]\,.$$

Extensions to a larger k-dimensional subspace are quite straightforward. In the following, we briefly present some applications of one-body and two-body SPF to phase transitions in CML.

The first example is spatiotemporal intermittency. A phase transition occurs from a laminar state to a turbulent state via an intermittently mixed region of the two, as a parameter is changed [#1] [1,2,4,5,9,10]. A simple example of spatiotemporal intermittency is given by a logistic map within the period-3 window (e.g., $a=1.752$) [1]. Both our one-body SPF (10) and direct numerical simulation give the same critical point $\epsilon\approx 10^{-2}$ for the transition from a laminar to turbulent states.

A simpler model for the intermittency is given by the choice of a piecewise-linear map [10]; $f(x)=ax$ ($x<\frac{1}{2}$), $f(x)=a(1-x)$ ($\frac{1}{2}<x<1$), and $f(x)=x$ ($x>1$). In the model, the motion is chaotic if $x<1$ and is regular (fixed point) for $x>1$. The CML corresponding to this $f(x)$ exhibits the spatiotemporal intermittency tran-

[#1] For a relevant experiment on spatiotemporal intermittency, see ref. [17].

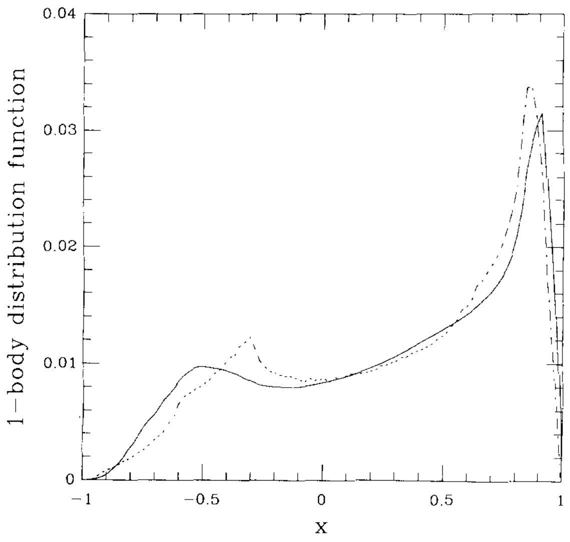

Fig. 1. One-body distribution function of $\rho(x)$. The solid line gives a 1-body distribution function obtained from a numerical integration of (10), while the dashed line gives that obtained from a direct simulation of (1). For the calculation of distribution, 100 mesh points are used for the interval $(-1, 1.1)$ ($\Delta x=0.021$). The logistic lattice with $a=1.95$ and $\epsilon=0.1$. For the direct simulation the size N is chosen to be 100 and a random initial condition is used.

sition at $\epsilon=\epsilon_c$. For $a=3.0$, the transition parameter is found to be $\epsilon_c \approx 0.36$ numerically. From the numerical integration of our one-body SPF (10), we have found $\epsilon_c=0.3333$. We have checked ϵ_c for a few different values of a, and both our SPF and the direct simulation agree rather well (within 10%). Our SPF solution gives a jump at the transition, that is, the measure for bursts given by $\int_{x<1} \rho(x)\,dx$ has a finite jump at ϵ_c.

Another example is the phase transition with pattern dynamics in the logistic map lattice [4]. The transition from the ordered pattern with some wavelength to a turbulent state is found, as a is increased. To see the ordered pattern with a domain size of l, we need an at least l-dimensional distribution function. Here, we have investigated the transition from a zigzag pattern ($l=2$) to a turbulent state for $\epsilon=0.1$. In fig. 2, two-point distribution functions $\rho(x(1), x(2))$ are shown. We can see a transition from a zigzag state to a turbulent state. This kind of transition is seen even in a two-coupled map [18]. In the treatment here the effect of other sites than the two-lattice-point subspace is included as a self-consistent heat bath.

In this Letter we have presented a simple formulation for a self-consistent Perron–Frobenius operator. The convergence to a fixed point function here is exponential and quite rapid (in our examples within 20 steps), while in the direct simulation, the covergence is $1/\sqrt{\text{time}}$ and requires more than 1000 steps.

If we take a larger k-dimensional subspace, it is expected that our result would be better. When spatial correlation decays exponentially, as is typically the case, our heat-bath procedure will be good if the subsystem size is larger than the correlation length.

Extensions of our formulation to the open-flow CML model [7], and higher-dimensional lattice [5] are straightforward (for a higher dimension, there are some difficulties [19], which may be resolved as in some cases of cellular automata [20]).

Also it may be possible to have a statistical mechanical argument for spatiotemporal chaos (see also ref. [11]), based on our PF operator for a subspace. Through this argument we hope to relate various quantifiers such as Lyapunov spectra [3], dimension density [12], co-moving and subspace-Lyapunov exponents [7,21], and mutual information flow [3].

Finally, we note that our formulation is *not* a mean-field theory. The mean-field theory in the original sense can be derived as a global coupling model for our lattice system, i.e.,

$$x_{n+1}(i) = (1-\epsilon) f(x_n(i)) + \frac{\epsilon}{N} \sum_j f(x_n(j)) .$$

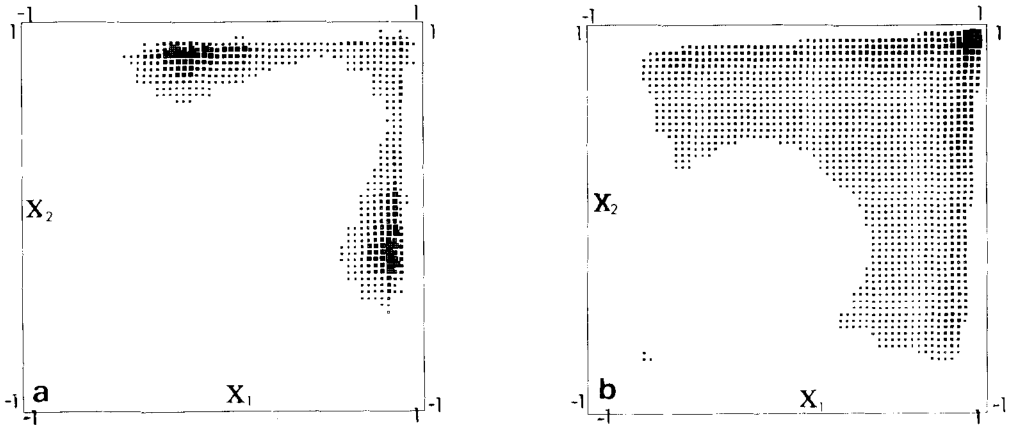

Fig. 2. Two-body distribution function of $\rho(x_1, x_2)$, obtained from the numerical integration of (11). For integration 64 meshes are used for $(-1, 1)$ $(\Delta x = 2/64)$. In the figure, a side of a square is proportional to $\rho(x_1, x_2)$ at the corresponding site. (a) $a=1.78$, $\epsilon=0.1$ (the maximum of $\rho(x_1, x_2)(\Delta x)^2$ is 0.007; the corresponding pixel is left blank if $\rho(x_1, x_2)(\Delta x)^2 < 0.0005$). (b) $a=1.98$, $\epsilon=0.1$ (the maximum of $\rho(x_1, x_2)(\Delta x)^2$ is 0.0035; the pixel is left blank if $\rho(x_1, x_2)(\Delta x)^2 < 0.00025$).

This equation has turned out to have a rich variety of phases corresponding to the pattern dynamics of our short-ranged lattice systems [22], as the mean-field model for a spinglass has given an interesting phase by Parisi [23].

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