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Diffusion in Hamiltonian dynamical systems with many degrees of freedom

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Hamiltonian dynamical systems with many degrees of freedom are investigated using symplectic map lattices. It is shown that anomalous diffusion exists only up to some crossover time beyond which the diffusion is normal. A diffusion constant, which is inversely proportional to the crossover time, exhibits a faster than any power-law dependence on the nonintegrability parameter, strongly suggesting the relevance of a bound by Nekhoroshev for Arnold diffusion. The motion in the standard mapping is also reexamined to show that the flicker noise is seen only down to some crossover frequency.

The understanding of the nature of Hamiltonian systems has progressed considerably in recent years.[1] The minimal-dimensional case has been investigated intensively through the "standard mapping."[2] The onset of global stochasticity with the collapse of the last Kolmogorov-Arnold-Maser (KAM) torus has been studied through the renormalization-group method, the diffusion process in the phase space has been investigated through the random-phase approximation in the highly nonintegrable regime, and the self-similar structure of islands and cantori and their relevance to anomalous diffusion and flicker noise have been clarified.[3–5]

The characteristic features of Hamiltonian systems with *many* degrees of freedom, however, are not well understood. It is thought that a process called "Arnold diffusion" connects the whole stochastic layer, thus the phase space of large Hamiltonian systems is topologically distinct from those of 2 degrees of freedom.[6] At the same time, however, for a weakly nonintegrable system, KAM tori still have a positive measure in the phase space. The former can be considered as the origin of statistical mechanics, since in the standard mapping with $K<K_c$, the phase space is separated by KAM tori, which destroy the ergodicity.

For a system of many degrees of freedom, it is expected that the small-scale structure found in the low-dimensional systems is smeared out, and that normal relaxation takes place, which assures the approach to a canonical ensemble within nonastronomical time.[7]

The difficulty in the study of the dynamics with many degrees of freedom lies in the very long-time scales (especially in weak nonintegrability) and which cannot be approximated perturbatively. The rate of diffusion is bounded by "Nekhoroshev's bound,"[6,8] which includes a singular factor like $\exp(-1/K^a)$ with K as a strength of perturbation to an integrable system and a is a positive constant.

There are few studies on the motion in phase space of Hamiltonian systems with many degrees of freedom. It is as yet unknown if Arnold diffusion is relevant to physics,[9] if the diffusion is normal or the motion includes the flickerlike noise, or if the diffusion rate has a singular dependence on the nonintegrability parameter as the above upper bound suggests. The necessity of long computation times for lattice *differential* equations has made it difficult to answer these questions. In the present paper we try to resolve these questions using a symplectic *map* lattice, to avoid this difficulty.

In Hamiltonian systems, the relevance of models with discrete time has been established in the standard mapping.[2] We extend this approach to lattice systems,[10–14] essentially following the idea of coupled map lattices.[15]

Here the following symplectic map lattice (SML) system is investigated:

$$\begin{aligned} x_{n+1}(i) &= x_n(i)+p_{n+1}(i)\,, \\ p_{n+1}(i) &= p_n(i)+(K/2\pi)(\sin\{2\pi[x_n(i+1)-x_n(i)]\} \\ &\quad +\sin\{2\pi[x_n(i-1)-x_n(i)]\})\,, \end{aligned} \tag{1}$$

where $i=1,2,\dots,N$ with periodic boundary conditions. The symplectic condition

$$\sum_i dx_n(i)\times dp_n(i)=\sum_i dx_{n+1}(i)\times dp_{n+1}(i)$$

is satisfied so that the model corresponds to a Hamiltonian system (where $\times$ denotes the exterior product). Here we discuss a one- (spatial) dimensional lattice only, but the extension to higher spatial dimensions is straightforward.

The dynamics (1) is obtained from the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{N}[p(i)]^2-[K/(2\pi)^2]\left(\sum_{i=1}^{N}\cos\{2\pi[x(i+1)-x(i)]\}\sum_{n=-\infty}^{n=\infty}\delta(t-n)\right), \tag{2}$$

or may be regarded as a Poincaré mapping from a lattice-differential equation. These interpretations are similar to those for the standard mapping.[1–3]

We focus our attention on two quantifiers: (1) the power spectrum, defined by

$$P(\omega)=\left\langle\!\left\langle\left|\sum_n p_n(j)e^{2\pi i n\omega}\right|^2\right\rangle\!\right\rangle;$$

and (2) the short-time diffusion coefficient, defined by

$$D(t)=\left\langle\!\left\langle(1/M)\sum_{m=1}^{M}(1/N)\sum_{i=1}^{N}[p_{t(m+1)}(i)-p_{tm}(i)]^2/t\right\rangle\!\right\rangle, \tag{3}$$

where $\langle\langle\cdots\rangle\rangle$ represents a temporal average. The summation over m in (3) is taken for the M sequential temporal average.

If the diffusion in phase space is normal, there exists a finite constant $D_\infty=\lim_{t\to\infty}D(t)$. If the diffusion is anomalous, $D(t)=t^{-\delta}$, with some exponent δ, which characterizes the stickiness of such diffusion.[16] Through a Fourier transform, it follows that the power spectra exhibit $P(\omega)\approx\omega^{-\alpha}$ with $\alpha=2-\delta$. If the diffusion is normal $\alpha=2$.

We calculate these two quantities for the SML (1) and for the standard mapping

$$p_{n+1}=p_n+(K/2\pi)\sin(2\pi x_n),\quad x_{n+1}=x_n+p_{n+1} \tag{4}$$

to study the characteristic motion in phase space in Hamiltonian systems.

First, we recall the standard mapping. It is believed that Hamiltonian systems with 1.5 or 2 degrees of freedom exhibit power-law decay of temporal correlations.[5] For the standard map, power spectra of the momentum look like $\omega^{-\alpha}$ for $\omega\to 0$, with some exponent $1<\alpha<2$. This $1/\omega$-type behavior has been understood as hierarchical diffusion threading through a self-similar island structure. Actually, $D(t)$ shows a power-law decay for some range of time interval t.

Although the picture of hierarchical diffusion seems plausible, one may expect that the self-similarity is not complete, especially for a system with many degrees of freedom. This logic leads one to expect the absence of such anomalous diffusion on very long time scales.

We have simulated the standard map starting from initial conditions belonging to the stochastic sea. The diffusion coefficient [Eq. (3) without the summation over the lattice sites i] exhibits power-law decay up to some t_c ($\approx 1/\omega_c$), with a power $t^{-\delta}$ ($\delta<1$). The diffusion constant D_∞ behaves as $(K-K_c)^{3.01\cdots}$, as is studied by the renormalization group, where $K_c(=0.97\ldots)$ is the parameter at which the last KAM torus breaks up.[3] In our numerical data, the inverse of the crossover time t_c is proportional to the diffusion constant D_∞. Corresponding to the diffusion coefficient, the power spectrum of the variable p (without taking "mod1") of length 2^{21} is calculated. We have found that, as $\omega\to 0$, the $\omega^{-\alpha}$-type behavior persists only down to a crossover frequency ω_c ($\approx 1/t_c$) and is replaced by ω^{-2}, which is reminiscent of simple (nonhierarchical) diffusive behavior ($1<\alpha<2$). The exponent α is consistent with the above relation with δ.

From these results we can conclude that anomalous diffusion resulting from the dynamical hierarchy holds only up to some crossover time. For longer times, the diffusion is normal if $K>K_c$. The crossover exists either because self-similar structure continues only down to some size except at $K=K_c$ or because the static self-similar structure ("fat" fractal[4]) does not reflect the dynamics of our quantity of such longer-time scales corresponding to very small-scale structures in phase space.[17]

Now let us proceed to the lattice system (1). Here, we have used a random initial condition. It turns out that the probability of hitting a KAM torus is very small if we do not choose a special class of initial condition [e.g., $x(i)=\text{const}$, $p(i)=\text{const}$], and the following results are independent of the choice of random initial conditions. For $K<1$, our model exhibits the sticky motion with a power spectrum of $\omega^{-\alpha}$ for a time interval not too long.[11] The crossover behavior is more easily seen in lattice models. As is expected from the Arnold diffusion in a system with many degrees of freedom, there is no barrier for the diffusion to infinity, if K is not zero. Neither the crossover frequency nor the diffusion constant shows singularity at any special value of K other than 0.

In Fig. 1, $D(t)$ is shown for some values of K. It shows the power-law behavior for $t_0<t<t_c$ (for $K<1$). Thus, the diffusion is anomalous in that time scale, reflecting the hierarchical structure in the phase space. For $t>t_c$, $D(t)$ approaches a constant, which means that the diffusion in our lattice system is normal even for small K. This suggests that the long-time behavior of generic Hamiltonian dynamical systems is well described by normal (nonhierarchical) diffusion. Again, t_c^{-1} is proportional to the diffusion constant D_∞ (Fig. 2).

Numerical results for the power spectra are consistent with this diffusion coefficient $D(t)$. For $K<1$, in the medium-frequency regime, they show $\omega^{-\alpha}$ with $1<\alpha<2$, but in the extreme low-frequency region they ap-

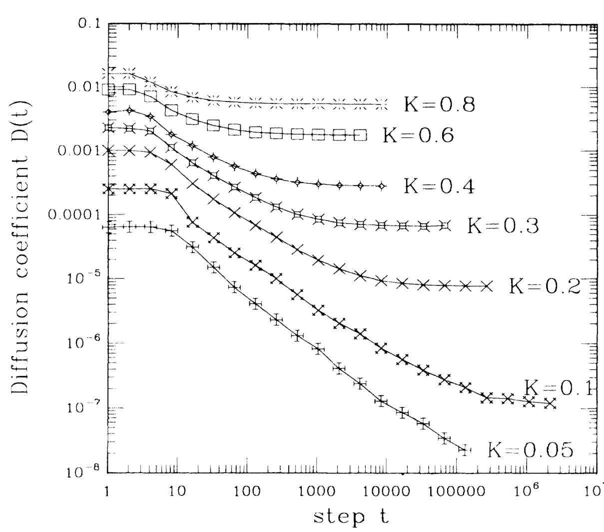

FIG. 1. A log-log plot of the short-time diffusion coefficient $D(t)$ for SML (1). $D(t)$ is calculated from 1000 sequential averages of t steps for $t\leq 2^{13}$, from 100 averages for $2^{13}<t\leq 2^{17}$, and 10 averages for $t>2^{17}$. System size $N=128$.

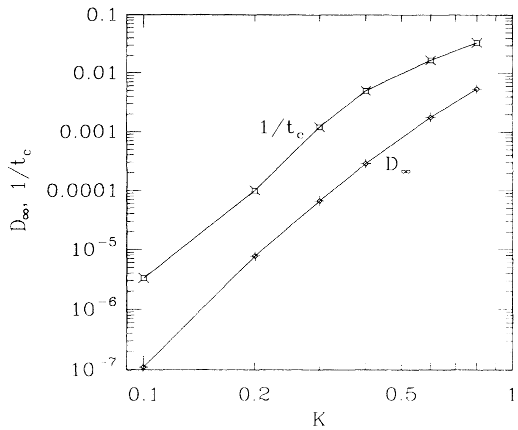

FIG. 2. A log-log plot of the diffusion constant D_∞ and inverse of crossover time t_c^{-1} vs K for SML (1). Calculated from the data in Fig. 1.

proach the normal ω^{-2} behavior. The estimated crossover frequency is the order of $1/t_c$, and the exponent α is again consistent with the relation $\alpha+\delta=2$.

The above results lead to the following picture: Up to some time scale, the motion in phase space "sticks" to some KAM tori, and the hierarchical structure of torus and islands leads to the power-law-type behavior. For longer time scales, the dynamics does not affect a small-scale structure, leading to a normal diffusion. One plausible reason for this is that the self-similar structure of islands and KAM torus is destroyed on the long-time scale for Arnold diffusion.

For $K>1.0$, our numerical results agree well with the prediction from a random update iteration which leads to $D_\infty = K^2$ and $P(\omega)=\omega^{-2}$.

As K goes to 0, the exponents δ and α slowly approach to unity (e.g., $\delta\approx 0.8$ for $K=0.1$ and $\delta\approx 0.9$ for $K=0.02$, and is close to 1 for $K=0.01$). Since the crossover time goes to infinity as K goes to zero, this means that the flickerlike noise with $1/\omega$ is generally observed in usual time scales in nearly integrable Hamiltonian systems.

The crossover time and diffusion constant are independent of the spatial lattice size (N) if N is larger than the spatial correlation length. This is assured by taking $N>8$ in our case. This size independence seems to be a general feature in locally interacting systems.[18]

What is the dependence of D_∞ on nonlinearity K? We can see that the shape in Fig. 2 is concave, which suggests that the singularity near $K\approx 0$ is higher than any power law. Nekhoroshev's bound leads to the form of $D_\infty < \exp(-\text{const}/K^a)$ (Ref. 2) with a constant a independent of N.[18] We have fitted our data by the form $D_\infty = \exp(-\text{const}/K^a)$, changing a. The fit is good if the exponent a is $0.1\sim 0.3$. With the present computer resources, it seems hard to confirm this specific form and to determine the value of a precisely, but our data suggest the singular dependence of diffusion constant on K near integrability, consistent with Nekhoroshev's bound.

To sum up, the diffusion in a Hamiltonian system is normal, but an anomalous diffusion is seen up to some time scale. The crossover to normal diffusion occurs at a scale inversely proportional to the diffusion constant. As the system approaches integrability, the diffusion constant goes to zero in a singular form as in Nekhoroshev's bound.

The relaxation rate to the equilibrium distribution turns out to agree with the diffusion constant.[11] Relations among other dynamical systems' quantifiers such as Lyapunov spectra will be discussed elsewhere.[19] The dynamical nature of Hamiltonian systems is thought to depend on the connectivity amongst the variables. Results of symplectic maps with long-range interaction and other connectivities will also be reported in the future.[19]

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