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On the Period-Adding Phenomena at the Frequency Locking in a One-Dimensional Mapping

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Frequency locking at the transition from torus to chaos is studied with the use of the map $\theta_{n+1}=\theta_n+0.25+A\cdot\sin(2\pi\theta_n)$ (mod 1). We find period-adding phenomena and various critical exponents, which are explained by extending Pomeau and Manneville's theory of intermittency.

In recent years, various routes to chaos and their critical phenomena have been extensively studied.[1)~4)] Previously[4)] we studied a 2-dimensional mapping and found a period-adding phenomenon at the frequency locking in the transition from torus to chaos. In this short note we show similar phenomena in a 1-dimensional mapping and give a simple explanation for the critical exponent.

We take the following map[5)]

$$\vartheta_{n+1}=F(\vartheta_n)\equiv\vartheta_n+D+A\sin(2\pi\vartheta_n). \quad (\mathrm{mod}\ 1) \qquad (1)$$

This map can be obtained by eliminating r_n through the condition $r_n=r_{n+1}$ in the map

$$\begin{cases} r_{n+1}=ar_n-g_1r_n^3+\tilde{A}\sin(2\pi\vartheta_n), \\ \vartheta_{n+1}=\vartheta_n+g_2r_n^2, \quad (\mathrm{mod}\ 1) \end{cases} \qquad (2)$$

or in the map

$$\begin{cases} r_{n+1}=(1-\gamma)r_n+d+a\sin(2\pi\vartheta_n), \\ \vartheta_{n+1}=\vartheta_n+r_{n+1}. \quad (\mathrm{mod}\ 1) \end{cases} \qquad (3)$$

We note that the map (2) is a discrete version of the complex Ginzburg-Landau equation with the additional perturbation $\tilde{A}$ $\sin(2\pi\vartheta_n)$ and the map (3) can be regarded as the extended standard mapping[3),6)] which includes the dissipation γ and the external pumping d. We fix the value D at 0.25 and

Table I. The stable period with the rotation number P/Q which appears at A. We change the parameter A by 5×10^{-5} for $0.12<A<0.1559$ in Table I(a) and by 10^{-5} for $0.1559<A<0.15633$ in Table I(b). (See the text for the definition of P and Q.)

(a)

$A\times10$	1.272	1.372	1.423	1.460	1.4815	1.5105	1.527	1.537	1.541
Q	9	23	14	33	19	24	29	34	73
P	2	5	3	7	4	5	6	7	15

$A\times10$	1.544	1.5465	1.5485	1.552	1.5545	1.5565	1.558
Q	39	83	44	49	54	59	64
P	8	17	9	10	11	12	13

(b)

$(A-0.156)\times10^4$	−0.8	0.2	1.0	1.4	1.7	2.3	2.8	3.0	3.2
Q	69	74	79	163	84	89	94	193	99
P	14	15	16	33	17	18	19	39	20

change A as a bifurcation parameter.

For $A<A_c=1/(2\pi)$ the map (1) is invertible and the attractor is a periodic orbit or torus. (We call the attractor 'torus' if the Lyapunov number equals 0 within the accuracy of 10^{-4}.) The rough phase diagram for $0\leq A\leq 0.25$ is shown in Fig. 1. Here, the rotation number is defined by $\lim_{n\to\infty}(F^n(\vartheta_0)-\vartheta_0)/n$, and when the attractor is Q_n-periodic orbit it is represented by P_n/Q_n, where P_n is the number of the times when the r.h.s. of (1) exceeds 1.0.

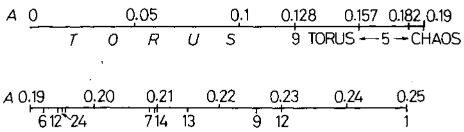

Fig. 1. Rough phase diagram of the map (1) for $D=0.25$. We change the parameter A by 0.01. The length of the period is written below the line. If any number is not written, the attractor is torus (for $A<1/(2\pi)$) or chaos (for $A>1/(2\pi)$).

As we increase A, the frequency locking at the rotation number $1/5$ occurs from torus and then a chaotic state appears. Thus, this mapping can be regarded as a simple model for the transition "Torus→Frequency Locking→Chaos", which was observed in various dissipative systems.[7)~10)]

As we change the parameter A, there appear a sequence of periodic orbits with the rotation numbers $P_n/Q_n=n/(5n-1)$, succesively at A_n and for $A>A_\infty=0.15671685\cdots$, the locking at $1/5$ occurs (see Table I).

In general, all rational numbers between p/q and r/s (p and q or r and s are relatively prime) are represented by $(np+mr)/(nq+ms)$ (n and m are integers).[9)] Thus, it is expected that there occurs a frequency locking at $(p+r)/(q+s)$ between the locking at p/q and r/s, since the rotation number is a continuous function of A. The period-adding sequence with the rotation number $n/(5n-1)$ can be understood as a locking between $(n-1)/(5(n-1)-1)$ and $1/5$. There also appears a frequency locking at $(2n-1)/(10n-7)$ between $n/(5n-1)$ and $(n-1)/(5n-6)$. (See Table I.) Here, we have to note that our numerical results suggest that the most stable period between the orbits with the rotation numbers p/q and r/s has the rotation number $(p+r)/(q+s)$. Though there exist infinite sequences (Farey sequences) of periodic orbits, we study only the $(5n-1)$-sequence in this short note, since it has a large stable region as will be seen later.

Next, we will study the convergence of A_n. We have obtained

$$A_\infty-A_n\propto n^{-2.0}\,, \qquad (4)$$

which is the same as the previous result of a 2-dimensional mapping.[4)] In order to study this critical phenomenon we plot $F^{(5)}(x)$ (mod 1) at $A=A_\infty$ in Fig. 2(a). As is seen from this figure, the locking at $1/5$ occurs via the tangent bifurcation. Thus, we can expand $F^{(5)}(x)$ around the periodic points $\{x_\nu^*\}(\nu=1,\cdots,5)$ for $A=A_\infty$ by

$$F^{(5)}(x)\approx(x-x_\nu^*)+a_\nu(x-x_\nu^*)^2+\varepsilon\,, \qquad (\text{mod } 1) \quad (5)$$

where ε is proportional to $(A_\infty-A)$. This form of expansion is common to the form of the theory of intermittency by Pomeau and Manneville.[2),11)] In our case, however, the state before the tangent bifurcation is not chaos but torus. From Eq. (5), we have

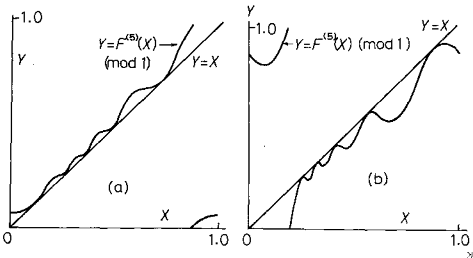

Fig. 2. $y=F^{(5)}(x\,;A)$ (mod 1) at
(a) $A=A_\infty$ (the onset of 5-periodic orbit),
(b) $A=A_c$ (the onset of chaos).

$$(\text{rot. no. at } A)-(\text{rot. no. at } A_\infty)\propto\sqrt{\varepsilon}\,, \quad (6)$$

since the time intervals while the orbit stays close to $\{x_\nu^*\}$ is proportional to $1/\sqrt{\varepsilon}$. The difference between rotation numbers at A_n and A_∞ is given by

$$n/(5n-1)-1/5=1/5(5n-1)\propto 1/n \quad \text{as } n\to\infty\,. \quad (7)$$

From Eqs. (6) and (7), we obtain $1/n\propto\sqrt{\varepsilon}$, that is $A_\infty-A_n\propto n^{-2}$.

In order to confirm our picture, we calculated numerically the minimum of the Lyapunov numbers $\lambda_n^{\min}$ for the $(5n-1)$-periodic orbit. For large n, we had $\lambda_n^{\min}\propto 1/n$. The Lyapunov number for $(5n-1)$-periodic orbit is given by

$$\lambda_n=\frac{1}{5n-1}\sum_{j=1}^{5n-1}\log|F'(x_j)|, \quad (8)$$

where $\{x_j\}$ are periodic points. We note that $\prod_{i=1}^5|F'(x_i)|\simeq 1$ for $\{x_i\}$ close to $\{x_\nu^*\}$, if n is large enough. According to our picture the number of such periodic points grows proportionally to $n(\propto\varepsilon^{-1/2})$, while the number of the points $\{\tilde{x}_k\}$ which are apart from $\{x_\nu^*\}$ is $O(1)$. Thus we have

$$\lambda_n\simeq\frac{1}{5n-1}\{O(n)\times\log 1+\sum_{\{\tilde{x}_k\}}\log|F'(\tilde{x}_k)|\}$$

$$\simeq\frac{1}{5n}\sum_{x_i\in\{\tilde{x}_k\}}|\log F'(x_i)|. \quad (9)$$

We expect that each periodic orbit has a similar structure and that the periodic points $\{\tilde{x}_k\}$ at $\tilde{A}_n$ are independent of n ($\tilde{A}_n$ is the value at which λ_n takes its minimum). Then $\sum_{x_i\in\{\tilde{x}_k\}}\log|F'(x_i)|$ is independent of n and we have $\lambda_n^{\min}\propto 1/n$.

Next, we study the self-similar structure of each $(5n-1)$-periodic orbit. The width $\Delta A_n\equiv A_n^f-A_n(A_n(A_n^f)$ is the value at which $(5n-1)$-periodic orbit gains (loses) its stability) is depicted in Fig. 3, which shows for sufficiently large n

$$\Delta A_n\propto n^{-3}\,. \quad (10)$$

Since $(A_{n+1}-A_n)\propto n^{-3}$ for large n, we have $\Delta A_n\propto(A_{n+1}-A_n)$, which shows a simple

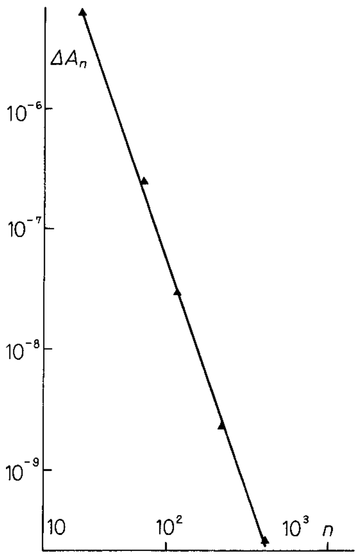

Fig. 3. $\log\Delta A_n$ as a function of $\log n$. We plot only $n=20$, 62, 122, 296 and 638. (See the text for the definition of ΔA_n.)

similar structure. It seems that $n\cdot\lambda_n(A_n+x\cdot n^3)$ is a universal function of x. This scaling may open the possibility of the renormalization group approach similar to the theory of intermittency.[11]

At $A=A_c=0.18189\cdots$, the 5-periodic orbit loses its stability and the chaotic state appears through the intermittency (see Fig. 2(b)). In the chaotic regime there appear successively the periodic orbits (windows) with the rotation number 1/6, 1/7, 1/8, $\cdots$, and at $A=0.25$ a stable fixed point appears. Thus, this mapping may become a simple model of the "periodic-chaotic transition", which was observed in B-Z reactions.[12]

In this short note, we have studied the period-adding structure of the frequency locking by extending the theory of intermittency. The period-adding structure is seen in various phenomena, such as B-Z reactions,[12] devil's staircases,[13] or windows of 1-dimensional mappings.[14] It will be interesting to study these phenomena from our picture. Frequency locking has been observed both in experiments[7),8)] and numerical integration of differential equations.[10)] Since our critical phenomenon seems to have a large universality class in 1- or 2-dimensional mappings, it may be expected that our theory is illustrated through more precise experiments. Detailed results with further studies on self-similarity and the effects of small noise will be reported elsewhere.

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