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Similarity Structure and Scaling Property of the Period-Adding Phenomena

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Period-adding sequence at the locking from torus to chaos is studied with the use of a one-dimensional mapping. Similarity and various scaling properties of each cycle are found numerically, which can be explained through a "phenomenological theory" based on the existence of the fixed point function. We classify the sequence into three cases. We also apply our theory to the case in which the sequence is a window among chaotic states.

§ 1. Introduction

In recent years, theoretical and experimental studies have been focused on the bifurcation to chaos in nonlinear systems. Especially, period-doubling bifurcations have formed a new paradigm based on the celebrated theory by Feigenbaum.[1] However, there is also an interesting phenomenon, namely a transition from torus to chaos, accompanied by the frequency locking at the rational values.[2] Recently Kadanoff and Shenker have studied the critical behavior of a KAM surface in area-preserving two-dimensional mappings.[3] In dissipative systems, however, theoretical studies are very few,[4] except the pioneering work by Ruelle and Takens.[5] We have studied a two-dimensional coupled-logistic map (that is, $x_{n+1}=1-Ax_n{}^2+D(y_n-x_n), y_{n+1}=1-Ay_n{}^2+D(x_n-y_n)$) and found a period-adding sequence at the frequency-locking and discussed its critical phenomenon.[6] In a previous paper,[7),*)] we studied the map[8),9)]

$$x_{n+1}=x_n+A\sin(2\pi x_n)+D \quad (\mathrm{mod}\ 1) \tag{1·1}$$

and found the same critical phenomenon. We also discussed its mechanism and related the problem of frequency locking to the theory of intermittency.[10] Since the intermittency has been studied extensively,[10),11)] we can understand the frequency locking through the study of intermittency.

In this paper we study the similarity of period-adding sequence in detail using the map (1·1). As we have already conjectured in a previous paper,[7] there is a similarity for the Lyapunov exponents of each periodic orbit. In § 2, we summarize the various critical phenomena and show the numerical results of similarity.

*) In Ref. 7), the term $\tilde{A}\sin(2\pi\theta_n)$ in Eq. (2) should read $\tilde{A}r_n\sin(2\pi\theta_n)$ and the left-hand side of Eq. (5) should read $F^s(x)-x_\nu{}^*$.

In § 3, we assume the 'scaling ansatz', which is justified through numerical results and we assume the existence of a fixed-point function. Since we cannot determine its form analytically as yet, we approximate it "phenomenologically" in a simple form, which seems to reproduce the numerical results well.

In § 4, we classify the period-adding sequences into three cases according to the forms of fixed-point functions and we illustrate them by numerical results for the map (1·1).

Period-adding sequence has also been found in various fields, for example, in the windows of a logistic map[12] and in the devil's staircase of the commensurate-incommensurate transition.[13] For Belousov-Zhabotinsky reaction, periodic-chaotic transition has been found theoretically[14] and experimentally.[15] In our map (1·1), this type of the transition has also been observed. We found

$$5 \to \text{chaos} \to 6 \to \text{chaos} \to \cdots \qquad \to 1 \tag{1·2}$$

as we increase A from 0.15 to 0.25 with the value D fixed at 0.25. This sequence obeys the same critical phenomenon. We study it in § 5 and discuss the similarity of each n-period.

Discussion will be given in § 6, where we comment on the universality of our critical phenomena and on the future problems such as the renormalization group approach.

§ 2. Critical phenomena of period-adding sequence

In this section we study the map (1·1) with D fixed at 0.25. As we described in the previous paper,[7] frequency locking at 1/5 (the notation of Q/P is the same as that in the previous paper) occurs at $A=0.1567168\cdots$ and this cycle loses its stability at $A=0.18189\cdots$ and chaotic state appears through a tangent bifurcation. We found that the cycle $n/(5n-1)$ appears at A_n and studied its critical phenomena in the previous paper.[7] We found $A_\infty - A_n \propto n^{-2}$, which was explained through the theory of intermittency. Since the locking occurs through a tangent bifurcation, this phenomenon is rather general. (Of course, there may appear the type of a tangent bifurcation with $x_{n+1}=x_n+\varepsilon+x_n^z$ with a general value of z instead of $z=2$ in our case. Then the relation that $A_n - A_\infty \propto n^{-z/(z-1)}$ follows and the values of various exponents to be described later will also change in a similar way.)

We also found $\Delta A_n \propto n^{-3}$($\Delta A_n \equiv A_n{}' - A_n$ and $A_n{}'$ is the value at which $(5n-1)$-cycle loses its stability) and that the minimum Lyapunov exponent obeys $\lambda_n^{\min} \propto n^{-1}$ (see Fig. 1). This suggests the following property that the scaled Lyapunov exponent as a function of A, defined by

$$\tilde{\lambda}_n(x) \equiv n\lambda_n(A_n + n^3 xc) \tag{2·1}$$

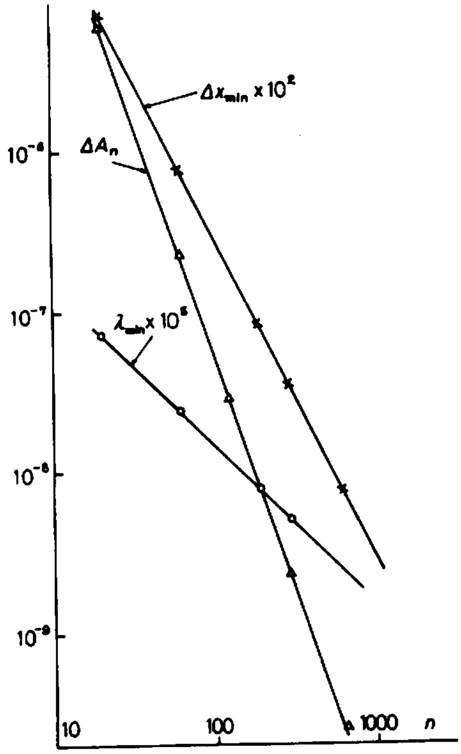

Fig. 1. Various scaling properties for the $(5n-1)$-sequence at $D=0.25$. The quantities $\log(\Delta A_n)$, $\log(\Delta x_n)$ and $\log \lambda_n^{min}$ are depicted versus $\log n$.

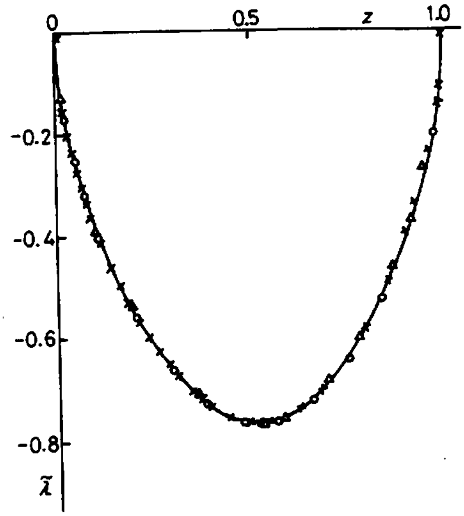

Fig. 2. The scaled Lyapunov exponents. The notations △, × and ○ show the exponents for $n=62$, 191 and 296, respectively. The line denotes the results of the phenomenological theory of §§3 and 4. (We take $\tilde{C}=0.54$.)

approaches a fixed point function $\tilde{\lambda}(x)$ as n goes to infinity. Here c is a constant, which we introduce in order to have $\tilde{\lambda}_n(0)=\tilde{\lambda}_n(1)=0$. We have plotted $\tilde{\lambda}_n(x)$ in Fig. 2 for $n=62$, 191 and 296. Thus, our prediction that $\tilde{\lambda}_n(x)\to\tilde{\lambda}(x)$ as $n\to\infty$ is verified. We note that $\tilde{\lambda}(x)\propto -x^{1/2}$ near $x\approx 0$ and $\tilde{\lambda}(x)\propto -(1-x)^{1/2}$ near $x\approx 1$, which was checked numerically. We cannot find the asymmetry about $x=0.5$. Here, we have to note that this function is not universal. As an illustration we considered the sequence with the rotation number $n/(5n-1)$ for $D\neq 0.25$. As we increase D, the value of A at which 5-cycle appears increases (it becomes larger than $1/2\pi$) and the function $\tilde{\lambda}(x)$ gets steeper and it becomes to take an infinite value (superstable cycle). (See § 4.)

Before proceeding to the detailed study, we show the scaling of Δx_n, which is the distance between the nearest periodic points of the $(5n-1)$-cycle. The quantity Δx_n obeys $\Delta x_n \propto n^{-2}$ as is given in Fig. 1. This scaling property will be important later.

We focus our attention on the structure of $f^{(5n-1)}(x)$ later in this section and

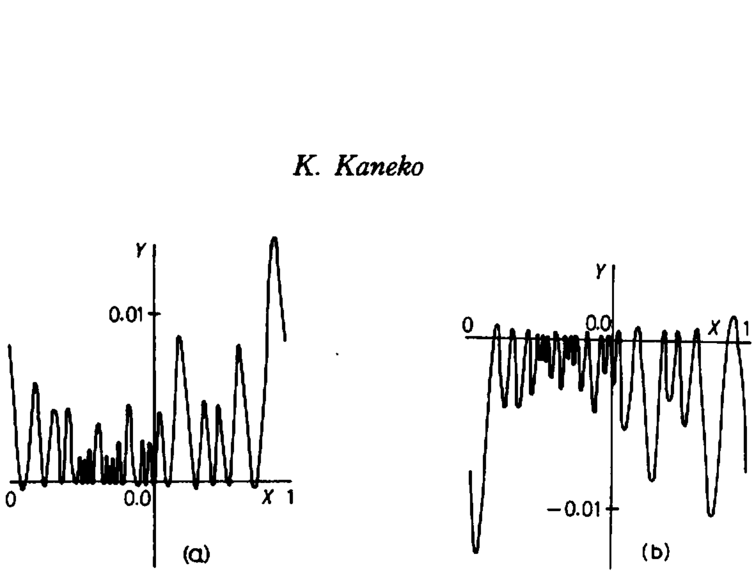

Fig. 3.(a) $f^{19}(x)-x$ at $A=0.14813$ and (b) at $A=0.14842$.

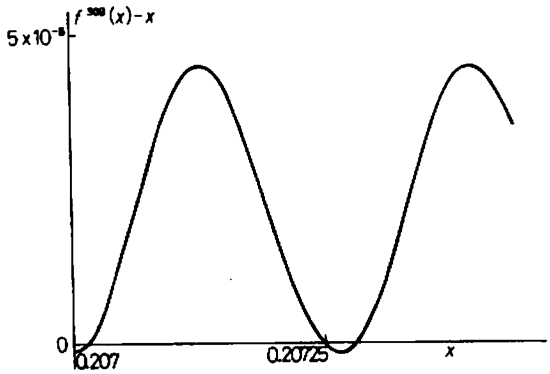

Fig. 4. $f^{309}(x)-x$ at $A=0.156673819$ for $0.2070<x<0.2074$.

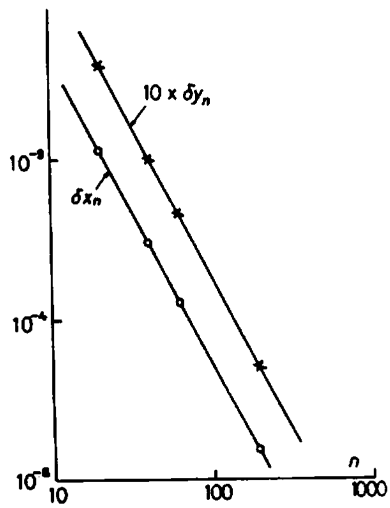

Fig. 5. The scaling properties of $f^{5n-1}(x)$. The notations $\times$ and $\circ$ denote $10\times\delta y_n$ and δx_n respectively.

in the next section. Here, $f(x)$ is defined by

$$f(x)=x+A\sin(2\pi x)+D. \quad (\text{mod } 1) \tag{2·2}$$

Since each $(5n-1)$-cycle appears via a tangent bifurcation, $y=f^{(5n-1)}(x)$ is tangent to $y=x$ at $(5n-1)$-points for $A=A_n$ and $A=A_n'$. (See Fig. 3 for $f^{19}(x)-x$ at $A\gtrsim A_{19}$ and $A\lesssim A_{19}'$.) As n increases by 1, the number of extremum points increase by 10. They appear near $\{x_\nu^*\}$ ($\nu=1, \cdots, 5$), where $\{x_\nu^*\}$ denotes the periodic points at A_∞.

Now we study the structure of $f^{(5n-1)}(x)$ near x_ν^*. (See Fig. 4 for $f^{309}(x)$ near $x=0.207$.) As is expected from the scaling of Δx_n, the quantity δx_n, which denotes the distance between the nearest extremum points $\underline{x}_n^\nu$ and $\underline{x}_n'^\nu$ near x_ν^*, obeys the scaling $\delta x_n \propto n^{-2}$. We note that the quantity $\delta y_n \equiv |f(\underline{x}_n'^{\nu}) - f(\underline{x}_n^{\nu})|$ also obeys the scaling $\delta y_n \propto n^{-2}$. (See Fig. 5.) This is a key to understand the similarity of Lyapunov exponents.

Before closing this section, we note that the scaling of Δx_n is easily obtained. We have for $A \approx A_\infty$

$$f^5(x\,;A) \approx x + a(x - x_\nu^*)^2 + \varepsilon \tag{2·3}$$

for $x \approx x_\nu^*$.[7] Since Δx_n is the distance between the nearest periodic points close to x_ν^*, it is given by $f^5(x_\nu^n) - x_\nu^n$, where x_ν^n is the periodic point nearest to x_ν^*. Using Eq. (2·3), we obtain $\Delta x_n \simeq \varepsilon \propto n^{-2}$. Thus, the scaling ($n^{-2}$) of Δx_n, δx_n and δy_n is expected from our picture based on the tangent bifurcation.

§ 3. "Phenomenological theory" of the similarity

In this section we give a simple frame to understand the results in § 2. We consider the case that the sequence of the cycle with the rotation number $(rn+s)/(pn-q)$ (that is, the locking between $(r+s)/(p-q)$ and r/p) appears at A_n (disappears at A_n^f) and at A_∞ the p-cycle appears. We follow the notations in § 2, since we always keep the case of § 2 ($p=5$, $q=r=1$, $s=0$) in mind in this section, but the results of this section hold in general.

Taking the results of Figs. 1 and 5 into account, we make the following scaling ansatz. Namely, concentrating only on the interval between $\underline{x}_n^{\nu}$ and $\underline{x}_n'^{\nu}$ (that is, the nearest periodic point near x_ν^*), we assume that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} a_n^{-1}(f^{pn-q}(a_n z + x_0^n\,;A_n) - x_0^n) = f^*(z). \tag{3·1}$$

We take x_0^n so that $f_n(0)=0$ and choose the scaling factor a_n so that $f_n(1)=1$ (that is, $y=f^{pn-q}(x)$ is tangent to $y=x$ at $x=x_0^n$ and $x=x_0^n+a_n$; we restrict our attention only to the interval $[x_0^n, x_0^n+a_n]$). Since $y=f^*(z)$ is tangent to $y=z$ at $z=0$ and $z=1$, it is convenient to define $g^*(z)$ by

$$g^*(z) \equiv f^*(z) - z \tag{3·2}$$

(see for example Fig. 4). The results in § 2 suggest that $a_n \propto n^{-2}$.

As we change A from A_n, the form of $f^{pn-q}(x)$ also changes. Since we confine ourselves only to the smallest structure of $f^{pn-q}(x)$, the main change is the addition of constants. Thus, we have approximately

$$f^{pn-q}(x\,;A) = a_n f^*(a_n^{-1}(x - x_0^n)) + x_0^n - \frac{1}{\gamma_n}(A - A_n) \tag{3·3}$$

for $A_n < A < A_n^f$ and $x_0^n < x < x_0^n + a_n$.

Numerical results show $\gamma_n \propto n^{-1}$, which can be easily understood as follows:

Using the chain rule, we have

$$f^{pn-q}(x\,;A_n+\delta A)-f^{pn-q}(x\,;A_n)=\frac{\partial f^{pn-q}(x\,;A_n)}{\partial A_n}\delta A$$

$$=\delta A\left\{\frac{\partial f^p(x_{p(n-1)-q};A)}{\partial A}\bigg|_{A_n}+\frac{\partial f^p(x_{p(n-2)-q};A)}{\partial A}\bigg|_{A_n}\frac{\partial f^p(x_{p(n-2)-q};A_n)}{\partial x_{p(n-2)-q}}\right.$$

$$+\cdots+\frac{\partial f^p(x_{p-q};A)}{\partial A}\bigg|_{A_n}\frac{\partial f^p(x_{p(n-2)-q};A_n)}{\partial x_{p(n-2)-q}}$$

$$\left.\times\frac{\partial f^p(x_{p(n-3)-q};A_n)}{\partial x_{p(n-3)-q}}\cdots\frac{\partial f^p(x_{p-q};A_n)}{\partial x_{p-q}}\right\}, \tag{3·4}$$

where x_k denotes $f^k(x)$. Since each term is independent of n as n gets large and the number of terms is n, we have

$$f^{pn-q}(x\,;A_n+\delta A)-f^{pn-q}(x, A_n)\propto n\delta A\,, \tag{3·5}$$

which means $\gamma_n\propto n^{-1}$. Though γ_n can depend on x in principle, we can approximate it as a constant, since x goes to $x_0{}^n$ as n gets large and α_n goes to 0.

Now, we consider the Lyapunov exponents. We can write

$$\lambda_n=\frac{1}{pn-q}\log|f^{(pn-q)\prime}(x_n{}^*)|\,, \tag{3·6}$$

where $x_n{}^*$ is an arbitrary fixed point of the map $x\to f^{pn-q}(x)$. We choose such a fixed point $x_n{}^*$ that satisfies $x_0{}^n<x_n{}^*<x_0{}^n+\alpha_n$. Using Eqs. (3·2) and (3·3), we have

$$\lambda_n=\frac{1}{pn-q}\log\{|1+g^{*\prime}(z_n{}^*)|\}, \tag{3·7}$$

where $z_n{}^*=\alpha_n^{-1}(x_n{}^*-x_0{}^n)$. The fixed point $x_n{}^*$ is given by

$$x_n{}^*+\alpha_n g^*(\alpha_n^{-1}(x_n{}^*-x))-\frac{1}{\gamma_n}(A-A_n)=x_n{}^*\,. \tag{3·8}$$

Thus we have $z_n{}^*=g^{*(-1)}((A-A_n)/\alpha_n\gamma_n)$, where $g^{*(-1)}$ is an inverse function of $g^*(z)$. The Lyapunov exponent is given by

$$\lambda_n(A)=\frac{1}{pn-q}\log\{|1+g^{*\prime}(g^{*(-1)}(\xi_n))|\}, \tag{3·9}$$

where $\xi_n=(A-A_n)/\alpha_n\gamma_n$. Thus we have

$$\tilde{\lambda}(z)=\frac{1}{p}\log\{|1+g^{*\prime}(g^{*(-1)}(z))|\} \tag{3·10}$$

as $n\to\infty$. Therefore, the existence of the fixed point function $g^*(z)$ means the existence of the fixed Lyapunov exponent $\tilde{\lambda}(z)$ and vice versa. Since $\alpha_n \propto n^{-2}$ and $\gamma_n \propto n^{-1}$, ξ_n is scaled by n^{-3}, which is consistent with the numerical result $\Delta A_n \propto n^{-3}$.

Up to now, we could not determine the form of g^* analytically. First we note that $g^*(0)=g^*(1)=0$ and $g^*(z)\propto z^2$, $g^*(1-z)\propto(1-z)^2$ for $z \ll 1$. We expand $g^*(z)$ by Fourier series. Taking only the lowest order and noting the above property of $g^*(z)$, we have

$$g^*(z)=C(1-\cos 2\pi z). \tag{3·11}$$

We take this simple form, because it gives an analytic expression for $\lambda_n(A)$ and a qualitatively correct result. The numerical results in § 2 (see Fig. 4) seem to support this simple approximation for the $(5n-1)$-sequence at $D=0.25$. Then we have $\sin^2 2\pi z^* = \tilde{\xi}_n(2-\tilde{\xi}_n)$ where $\tilde{\xi}_n = \xi_n/C = (A-A_n)/(C\alpha_n\gamma_n)$ and we obtain

$$\lambda_n(A)=\frac{1}{pn-q}\log\{|1-2\pi C\sqrt{\tilde{\xi}_n(2-\tilde{\xi}_n)}|\} . \tag{3·12}$$

There are three possibilities of the behavior of $\lambda_n(A)$ according to the value of $\tilde{C} \equiv 2\pi C$.

Before going to the detailed discussion, we comment on the properties of Eq. (3·12):

a) $\lambda_n(A)$ is scaled by $\alpha_n\gamma_n \propto n^{-3}$.
b) $\lambda_n(\tilde{\xi}_n)\propto -\tilde{\xi}_n^{1/2}$ and $\lambda_n(2-\tilde{\xi}_n)\propto -(2-\tilde{\xi}_n)^{1/2}$ for $\tilde{\xi}_n \ll 1$.
c) $\lambda_n(\tilde{\xi}_n)$ is symmetric with respect to $\tilde{\xi}_n=1$.

We note that properties a) and b) are not dependent on the approximate choice (3·11) of $g^*(z)$. The property c) is valid if g^* is symmetric.

We discuss three possible cases in the next section.

§ 4. Classification of period-adding sequences

As was discussed in § 3, the similarity of period-adding sequence can be understood through $g^*(z)$. There are three possible cases according to the curvature of $g^*(z)$:

Case I Each $(pn-q)$-cycle does not have a superstable one and loses its stability through a tangent bifurcation.

Case II Each $(pn-q)$-cycle has two superstable cycles and loses its stability through a tangent bifurcation.

Case III Each $(pn-q)$-cycle loses its stability through period-doubling bifurcations. (That is, the Floquet multiplier changes from $+1$ to -1.)

If the original map is invertible, there is no superstable cycle and the possible case is only I. Thus we have Case I for the period-adding sequence for the map

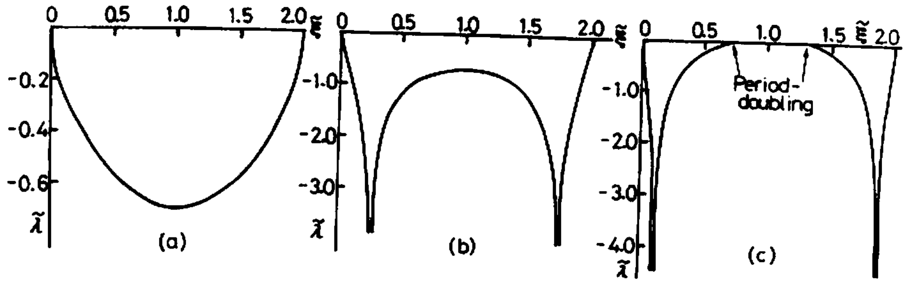

Fig. 6. The scaled Lyapunov exponent obtained from the phenomenological theory. The function $\log\{|1-\tilde{C}\sqrt{\tilde{\xi}(2-\tilde{\xi})}|\}$ (see Eq. (3·12)) for (a) $\tilde{C}=0.5$, (b) $\tilde{C}=1.5$, (c) $\tilde{C}=2.1$ are depicted.

(1·1) with $A<1/(2\pi)$.

We study these three cases using the function $g^*(z)=C(1-\cos(2\pi z))$ (see Eq. (3·11)). Three cases are classified by the value $\tilde{C}\equiv 2\pi C$. We have plotted $\tilde{\lambda}(\xi)$ given by Eq. (3·10) for these three cases.

Case I $0<\tilde{C}<1$

This condition corresponds to the case that the map is invertible. Thus, the attractor is torus or frequency locking. The Lyapunov exponent behaves like Fig. 6(a). This exponent has a minimum at $\tilde{\xi}_n=1$, and $\lambda_n^{\min}(A)\propto 1/(pn-q)\propto 1/n$ for large n. This is the case for the $(5n-1)$-sequence which was discussed in § 2.

We can fit the value $\tilde{C}$ from Fig. 5 for this case. Using this value and Eq. (3·12), we can plot $\lambda_n(A)$, which is shown in Fig. 2, which agrees well with the numerical results. We note that the Schwarzian condition is broken both for the map with $A<1/(2\pi)$ and $f^*(x)$ with $\tilde{C}<1$.

Case II $1<\tilde{C}<2$

The $(pn-q)$-cycle has two superstable cycles at $\tilde{\xi}_n=1-\sqrt{1-1/\tilde{C}^2}$ and at $\tilde{\xi}_n=1+\sqrt{1-1/\tilde{C}^2}$. It loses its stability at $\tilde{\xi}_n=2$ via a tangent bifurcation. The Lyapunov exponent behaves like Fig. 6(b).

As an example of this case we consider the map (1·1) with $D=0.253$. For this case the locking at $1/5$ occurs at $A_{\rm II}=0.161625\cdots(>1/2\pi)$. The sequence with the rotation number $n/(5n-1)$ appears for $A<A_{\rm II}$. The Lyapunov exponent for this sequence behaves just like Fig. 6(b). The approximate value of $\tilde{C}$ which we estimate from $f^{5n-1}(x)$ is 1.9. Thus this is an example for Case II. We note that there exists a chaotic region between $(5n-1)$-cycle and $(5n+4)$-cycle since A is larger than $1/2\pi$.

Case III $\tilde{C}>2$

The $(pn-q)$-cycle loses its stability at $\tilde{\xi}_n=1-\sqrt{1-4/\tilde{C}^2}$ at which the Floquet multiplier crosses -1 and period-doubling bifurcations occur. At $\tilde{\xi}_n=1+\sqrt{1-4/\tilde{C}^2}$, the $(pn-q)$-cycle restores its stability via inverse period-doubling bifurcations. If $\tilde{C}$ is large enough and the interval $[1-\sqrt{1-4/\tilde{C}^2}, 1+\sqrt{1-4/\tilde{C}^2}]$ is long, the period-doubling cascade reaches chaos. If $\tilde{C}$ is not large enough, the period-doubling cascade stops at some order and the period gets half by half.[16),*)] (That is, the bifurcation "$(pn-q)\to 2\cdot(pn-q)\to\cdots\to 2^{k-1}(pn-q)\to 2^k(pn-q)\to 2^{k-1}(pn-q)\to\cdots\to(pn-q)$" occurs as we increase the value of A.) We can roughly estimate the condition on whether the period-doubling reaches chaos, using Feigenbaum's δ.

Let us consider the example of Case III. As we increase the value of D from 0.254, the $(5n-1)$-sequence loses its stability through period-doubling bifurcations. For $D=0.254$, the period-doubling bifurcation stops at some order. We have observed such sequences as $119\to 238\to 476\to 952\to 476\to 238\to 119$ $(n=24)$ or $234\to 468\to 936$ $1872\to 936\to 468\to 234$ $(n=47)$ for example. For larger values of D, period-doubling bifurcations go to chaos. Here, we note that the inverse cascade for $(5n-1)$-sequence ("chaos$\to\cdots\to 2\cdot(5n-1)\to(5n-1)$") can appear after the period-doubling cascade for the $(5m-1)$-cycle, where m is larger than n. At $D=0.256$, for example, the cascade "$(5n-1)\to 2\cdot(5n-1)\to\cdots\to$chaos" appears before the cascade "chaos$\to\cdots\to 2\cdot(5(n-2)-1)\to 5(n-2)-1$" for large n as we increase the value of A. We also note that the period-doubling cascade for $(5m-1)$-cycle and the inverse cascade for $(5n-1)$-cycle $(m>n)$ can appear simultaneously at the same value of the parameter A. Thus the attractor is divided into multibasin. We have observed this phenomenon by changing the values of A and D.

We note that the Schwarzian condition is satisfied for the function $f^*(x)$ corresponding to Eq. (3·11) with $\tilde{C}>1$ (Cases II and III). Thus, the period-doubling cascade in Case III always obeys the theory of Feigenbaum,[1)] if the doubling goes to infinity.

Thus, all the three cases appear for the map (1·1). We note that this classification is independent of the special choice of $g^*(z)$ (Eq. (3·10)). For the sequence which appears at large A, the form (3·10) does not seem to be good, but the scaling and similarity hold just like in § 2. We give a typical example in the next section.

§ 5. Periodic-chaotic transition near $A=D$

In our map (1·1), we have found the period-adding sequence (1·2) as A approaches $D=0.25$, at which a stable fixed point appears via a tangent bifurcation. We study this sequence $(5\to 6\to 7\to 8\to\cdots)$. Thus, this is an example of periodic-chaotic transitions observed in Belousov-Zhabotinsky reactions.[14),15)] We have plotted in Fig. 7 the attractor for the map (1·1) with $D=0.25$ and for 0.18

*) The stop of the period-doubling cascade was first pointed out by Tsuda[16)] when the Schwarzian condition is not satisfied. (In our case, it is satisfied.)

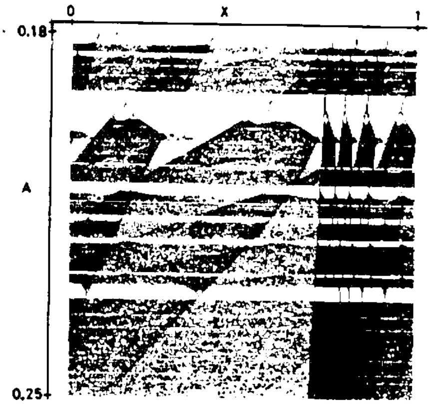

Fig. 7.

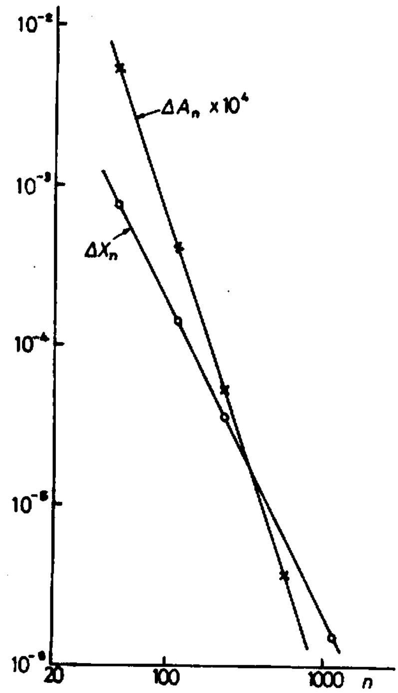

Fig. 8.

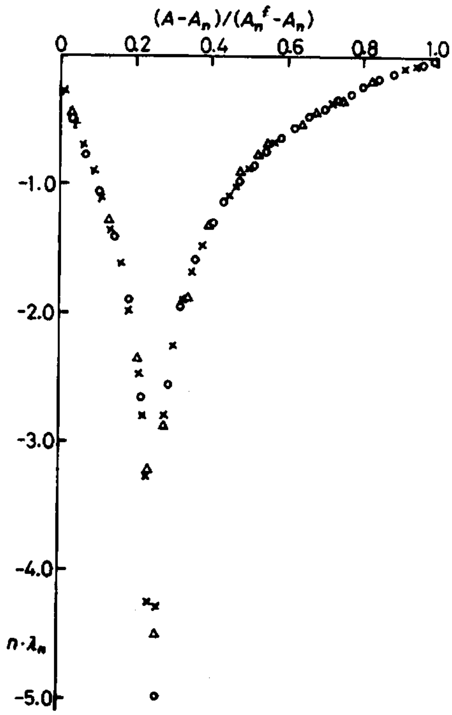

Fig. 9.

Fig. 7. The attractor for the map (1·1) with $D=0.25$ as we change A from 0.18 to 0.25. We have plotted x_ns for $500<n<1300$ for each A.

Fig. 8. Scaling properties of the n-sequence near $A \lesssim D=0.25$. The quantities $\log(A_\infty - A_n)$ (which coincides with $\log \Delta x_n$ within the numerical accuracy) and $\log \Delta A_n$ versus $\log n$ are depicted.

Fig. 9. Scaled Lyapunov exponent for the sequence with the rotation number $1/n$. The Lyapunov exponent is scaled by n^{-1}, while $(A-A_n)$ is scaled by n^{-3}. We have plotted the results for $n=63$(○), 118(×) and 554(△).

$<A<0.25$. Though the 5-period cycle changes to chaos via intermittency, the n-cycle ($n>5$) goes to chaos via period-doubling bifurcations. Thus, this is Case III in § 4.

First, we have checked $A_\infty - A_n \propto n^{-2}$ (see Fig. 8), which agrees with our theory. Since the tangent bifurcation at $A=0.25$ is of the type "$x_{n+1}=x_n+ax_n^2+\varepsilon$" (type I intermittency with $z=2$[10]), this is rather obvious. We have also checked $\Delta A_n \propto n^{-3}$ and $\Delta x_n \propto n^{-2}$. We note that $\Delta x_n = A_\infty - A_n$, since $\varepsilon = A_\infty - A_n$ holds in this case. (Since the superstable point exists, $\lambda_n^{\min}$ does not exist.) The scaled Lyapunov exponent behaves like Fig. 6(c), as is shown in Fig. 9, which seems to show the fixed Lyapunov exponent.

The function $f^n(x)-x$, however, is very complicated even if we look only at the interval between the nearest periodic points. It has a lot of extremum points in the interval and the simple choice of the function $f^*(x)$ corresponding to Eq. (3·11) is impossible. This may be due to the fact that A is large and the non-invertible regime of the function $f(x)$ is wide. Since the scaling properties are valid, some modifications of §§ 3 and 4 may be possible for this case.

§ 6. Discussion

In this paper we have discussed the similarity of the period-adding sequences. Since various scaling properties and phenomenological theory are based on the fixed-point function $g^*(z)$, it will be possible and necessary to construct a renormalization group approach.[1),3),11)] It has some difficulties, since the function $g^*(z)$ is not universal at all and the scaling factor α_n is not α^n but n^{-2} and the n-independent recursion relation is not at hand. This study is now in progress.

In this paper, we have confined ourselves to the one-dimensional map (1·1). Our theory in §§ 3 and 4, however, is rather general. It can also be applied to the two-dimensional mappings[6),17)] or differential equations, by projecting onto one-dimensional space of the angle "x_n". Recently Sano and Sawada[18)] have found the period-adding sequence and checked that $(A_\infty - A_n) \propto n^{-2}$ for the coupled-Brusselator model (differential equations with four variables). It will be interesting to search for this phenomenon in experiments such as Bénard or Taylor problems.[2)]

As to the observation of frequency locking, we have to note that any locking with the rotation number $(nq+ms)/(np+mr)$ appears in principle between the lockings with the rotation numbers q/p and s/r. Whether it is feasible to be observed or not depends on the regions of the parameter where the cycle is stable (i.e., the size of ΔA_n) and the stability of the cycle (i.e., λ_n). Since ΔA_n is scaled by n^{-3} and λ_n by n^{-1}, the shorter cycle (i.e. smaller n) is easier to be observed. This result about the observability for shorter cycles seems to be rather general, since γ_n^{-1} is proportional to the period (see Eqs. (3·3) and (3·4)). In this sense, the most stable sequence between the cycles with the rotation numbers with $1/(p-1)$ and $1/p$ ($p=5$ for § 2) is the sequence with the rotation number $n/(pn-1)$. For dissipative systems, therefore, the period-adding sequence will be easier to be observed than the Fibonacci sequence.[16),8)]

As for the related problems for the conservative systems, we have to note that the devil's staircase of commensurate-incommensurate transition[13)] may be related to the theory of intermittency for area-preserving maps,[19] similarly to the present argument for dissipative mappings.

When we come back to the map (1·1), there are many problems left for future, such as the effect of small noise, the similarity of chaotic regions between periodic orbits, and the mechanism of the development of chaos for $A>1/(2\pi)$. These problems will be reported elsewhere.

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