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Doubling of Torus*)

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"Doubling of torus" is found in three-or four-dimensional dissipative mappings. In these models, doubling occurs only a finite number of times before chaos appears. An explanation is attempted by a numerical study on the stablility against perturbation of the direct product state of coupled logistic and torus maps.

Transition from torus to chaos in dissipative systems has been extensively studied.[1)~10)] Phase motion on a torus has been investigated by the one-dimensional map[2)~7)] $\theta_{n+1}=\theta_n+\frac{K}{2\pi}\sin(2\pi\theta_n)+\Omega(\mathrm{mod}\ 1)$, where various critical phenomena among locking states have been found. Inclusion of the amplitude motion has been considered in two-dimensional mappings,[8)] where the distortion of torus is found, which is explained in connection with the oscillation of an unstable manifold of a periodic saddle.[9)] These maps can be regarded as a Poincaré plot (or just the phase motion part of it) of three-dimensional flow.

What happens to a torus in a higher dimensional flow? In this paper, we report the discovery of "doubling of torus" in dissipative mappings. Since more than three variables are necessary for the doubling of torus in a flow system, the Poincaré plot includes at least three variables for the system with no conservation law. Therefore, we study three- or four-dimensional mappings to search for this phenomenon.

Before proceeding to show the specific maps, we note that two types of doubling of torus are possible for mappings, that is, the case in which the cross section of a torus is separated (see Fig. 1(a)) (type a) and the case in which it is still connected but two-fold (see Fig. 1(b)) (type b).

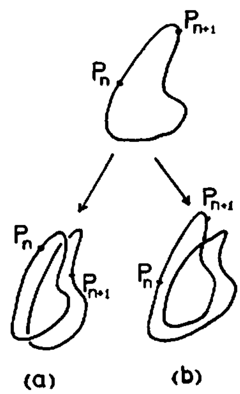

Fig. 1. Schematic illustration of two types of doubling of torus in k-dimensional mappings ($k\geq 3$) $P_n\to P_{n+1}$.

The maps investigated here are

(I) $X_{n+1}=AX_n+(1-A)(1-DY_n^2)$,
$Y_{n+1}=Z_n$, $Z_{n+1}=X_n$; $A=0.4$,

(II) $X_{n+1}=AX_n+(1-A)(1-DY_n^2)$,
$Y_{n+1}=Z_n$,
$Z_{n+1}=W_n$, $W_{n+1}=X_n$;
$A=0.3$ or $A=0.4$,

(III) $X_{n+1}=AX_n+(1-A)(1-DY_n^2)$,
$Y_{n+1}=AY_n+(1-A)(1-DZ_n^2)$,
$Z_{n+1}=AZ_n+(1-A)(1-DX_n^2)$;
$A=0.4$,

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and

(IV) $X_{n+1}=AX_n+(1-A)(1-DY_n^2)$,
$Y_{n+1}=Z_n$,
$Z_{n+1}=AZ_n+(1-A)(1-DW_n^2)$,
$W_{n+1}=X_n$; $A=0.3$,

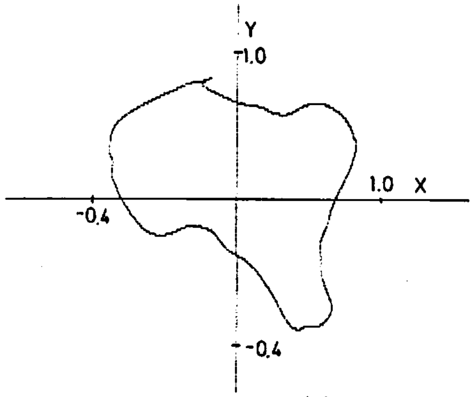
where D is changed from 1.0 to 2.2 as a bifurcation parameter and A is fixed at the listed values.

These maps show the transition "fixed point → torus → doubling of torus → chaos", with frequency lockings intervening. The doubling is of type (a) for map (I), while it is of type (b) for maps (II)~(IV).

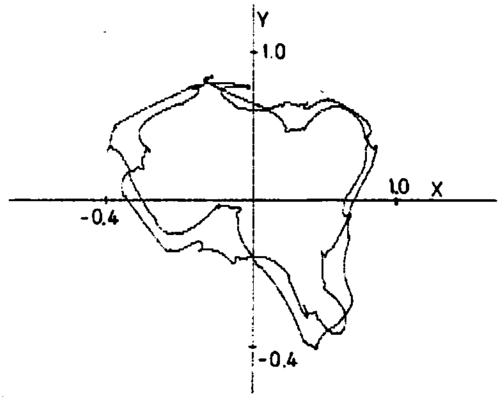
Some examples of the attractors are given in Figs. 2(a)~(c) (for map (I)) and in Figs. 3(a)~(c) (for map (IV)), where projections onto (X, Y)-plane are depicted. Lyapunov exponents are calculated to confirm the successive bifurcations. As an example, the first and second Lyapunov exponents for map (IV) are shown in Fig. 4 (the third and fourth exponents do not change drastically and they are omitted in the figure).

The parameters at which the doubling occurs are given in Table I. As is shown in this table, the doubling cascade stops after a finite number of times and the system undergoes a transition to chaos (torus → 2×torus → 4×torus → chaos etc.).

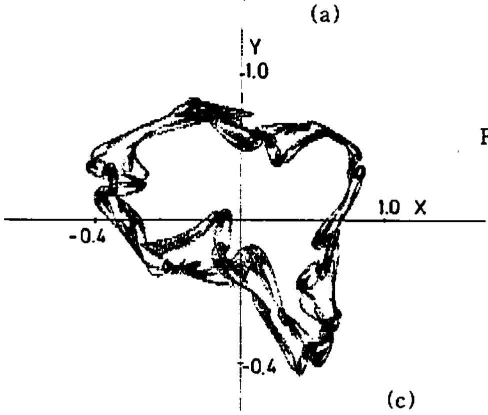
Thus, there arises a problem, whether the doubling cascade of torus can continue infinitely in general or not. Following essentially the idea in the celebrated paper of Ruelle and Takens,[1] we simplify and restate this problem as follows: Is the direct product state of the torus map (e.g., $Y_{n+1}=Y_n+C$ (mod 1)) and the map which obeys Feigenbaum's theory[11] (e.g., $X_{n+1}=1-AX_n^2$)



(a)



(b)



(c)

Fig. 2. Projection onto (X, Y)-plane of the attractor of map (I) with $A=0.4$.
(a) $D=2.11$. (torus)
(b) $D=2.16$. (2⊗torus)
(c) $D=2.19$. (chaos)
If we take only the points (X_{2n}, Y_{2n}) in Fig. 2(b), only one torus remains. Thus, this doubling is of type (a).

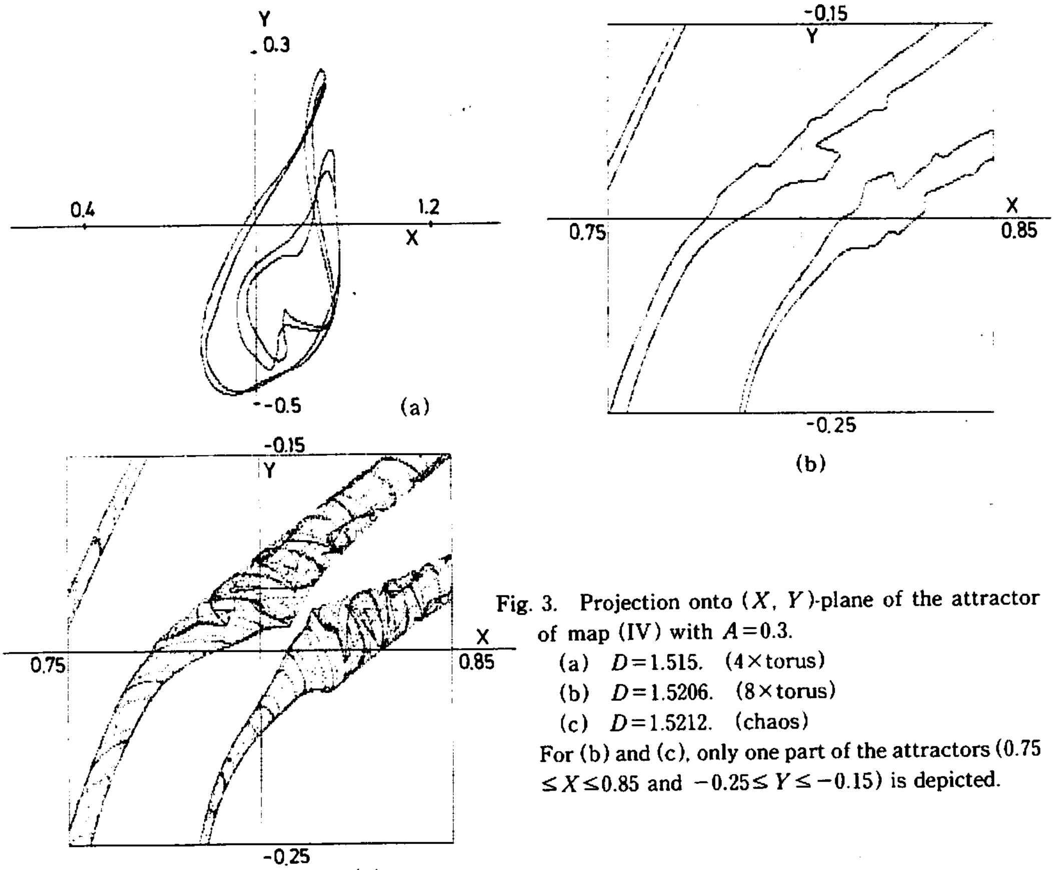


Fig. 3. Projection onto (X, Y)-plane of the attractor of map (IV) with $A=0.3$.
(a) $D=1.515$. ($4\times$torus)
(b) $D=1.5206$. ($8\times$torus)
(c) $D=1.5212$. (chaos)
For (b) and (c), only one part of the attractors ($0.75 \le X \le 0.85$ and $-0.25 \le Y \le -0.15$) is depicted.

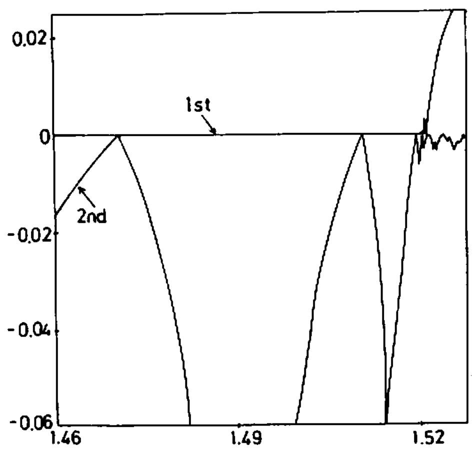


Fig. 4. The first and second Lyapunov exponents for map (IV) with $A=0.3$. We made 50000 iterations for calculations, with double precision.

stable against a perturbation or not? We performed a numerical simulation of the map

$$\begin{cases} X_{n+1}=1-AX_n^2+\varepsilon g(X_n, Y_n), \\ Y_{n+1}=Y_n+C+\varepsilon h(X_n, Y_n), \quad (\text{mod } 1) \end{cases} \tag{1}$$

where g and h are perturbations chosen to be $g(X_n, Y_n)=\sin(2\pi Y_n)$ and $h(X_n, Y_n)=X_n$ for simplicity*) and the parameter C is fixed at $(\sqrt{5}-1)/2$, i.e., the inverse of golden mean. When $\varepsilon=0$, variables X and Y are decoupled and the transition "torus $\rightarrow 2\otimes$torus $\rightarrow 4\otimes$torus $\rightarrow \cdots \rightarrow$ chaos" proceeds as A is increased. From the simulations up to ε

*) When $h\equiv 0$, the problem reduces to the logistic map with incommensurate perturbation.[2]

Table I. The parameter values at which the doubling occurs and the value of the onset of chaos for models (I)~(IV). The values D_k denote the parameter values at which the doubling $2^{k-1}\times$torus $\to 2^k\times$torus occurs, while the values D_c denote the onset of chaos. The times of doubling observed before the onset of chaos are also shown. These values are obtained from the calculations of Lyapunov exponents and the graphs of the attractors.

model	D_1	D_2	D_3	D_c	times of doubling
(I) $A=0.4$	2.151			2.163	1
(II) $A=0.3$	1.539			1.62	1
(II) $A=0.4$	1.694	1.90409		1.90455	2
(III) $A=0.4$	1.740			1.941	1
(IV) $A=0.3$	1.470	1.5106	1.5199	1.5209	3

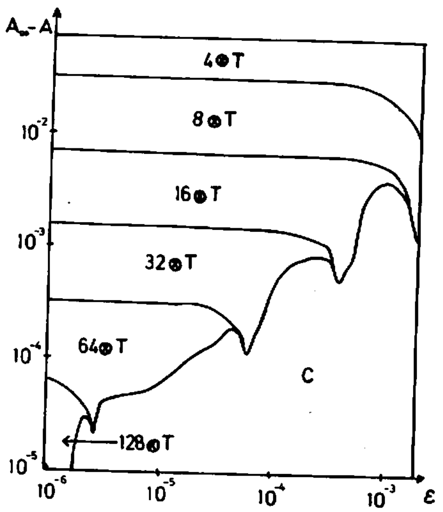


Fig. 5. The phase diagram of map (1). The longitudinal axis denotes $(A_\infty - A)$, where A_∞ ($=1.4011551\cdots$)[11] is the value of the onset of chaos for the map $x_{n+1}=1-Ax_n^2$. The transverse axis denotes ε. This diagram is obtained through calculations of the first and second Lyapunov exponents and the graphs of the attractors. "C" and "$n\otimes T$" represent chaos and $n\otimes$torus respectively.

$=10^{-7}$, however, it has been found that the doubling cascade stops after a finite number of times for finite ε and the chaos appears from the state $2^l\otimes$torus, where l is a finite integer, which increases as ε approaches zero. A rough phase diagram is given in Fig. 5, which was obtained from the calculations of Lyapunov exponents and from the patterns of the attractors. The shape of the torus is distorted[9] before the onset point of chaos as is the case of maps (I)~(IV) (see Fig. 3(c)).

Thus, the direct product state of the torus and the doubling becomes unstable by the introduction of the above perturbation. This result gives a qualitative explanation to the conjecture that the doubling of torus stops after a finite number of times generically. Figure 5 reminds us of the bifurcation gap, found by Crutchfield and Huberman[13] for the period-doubling in the presence of a random noise. Detailed results will be reported elsewhere, including the scaling relation[14] between the perturbation ε and the number of doublings l, with some theoretical considerations.[14],[15] The doubling of torus reported in this paper will be found experimentally, if it is observed that the peak positions of the power spectrum show the change "$n\omega_1+m\omega_2 \to n\omega_1+m\omega_2/2 \to \cdots \to n\omega_1+m\omega_2/2^k \to$ continuous spectrum" ($\omega_1/\omega_2=$irrational, $k=$integer, and $n, m=0, 1, 2, \cdots$). It will be of interest to search for such a phenomenon in the Bénard or Taylor problems.

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Note added: After this work was submitted, Phys. Letters **94A** arrived at the author, where he found the paper "Cascade of Period Doublings of Tori; A. Arnédo, P. H. Coullet and E. A. Spiegel, Phys. Letters **94A** (1983), 1", which reported the discovery of the doubling of tori in a three-dimensional mapping.