Fractализация тороида

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Для определенного класса двухмерных отображений, тороид протекает более сложным образом, чем бифуркационный параметр увеличивается. Это становится фрактальным в критической точке и хаос появляется выше этой критической точки. Открытие этого явления, вычисление фрактальной размерности и универсальности этого явления сведено в этом документе.

Various nonequilibrium systems show transition from torus to chaos. Its mechanism and critical phenomena have extensively been studied. Especially, the instability of phase motion has been analyzed in detail using a one-dimensional circle map. The amplitude behavior of torus, however, has been less investigated. In this letter, we study the oscillation of torus, which was observed in various two-dimensional mappings and also in experiments. In usual two-dimensional mappings, however, the region of frequency lockings increases near the onset of chaos, which masks the oscillatory behavior.

To study the oscillatory behavior in more detail, we introduce here a model which has no locking by its construction. It is a modulation map given by

\[ x_{n+1} = f(x_n) + \varepsilon h(\theta_n), \]
\[ \theta_{n+1} = \theta_n + c, \quad (\text{mod 1}) \] (1)

where \( c \) is fixed at an irrational number, thus, no locking appears and \( h(\theta) \) is a periodic function of period 1. This type of mapping has also been studied in the doubling of torus and in the problem of three-torus. We take here \( f(x) = ax + bx^2 \) and \( h(\theta) = \sin(2\pi \theta) \), and consider the case that there is a stable torus for small \( \varepsilon \). For \( \varepsilon \approx 0 \), the torus is almost straight. As \( \varepsilon \) is increased, it oscillates more and more strongly till it collapses at some critical value \( \varepsilon = \varepsilon_c \) and chaos emerges (see Fig. 1a, b, and c) for the attractors. The value \( \varepsilon_c \) is numerically confirmed by the calculations of Lyapunov exponents.

In order to investigate the oscillations of torus in more detail, we study the equation for the invariant curve. The invariant curve \( x = g(\theta) \), if it exists, must obey the functional equation

\[ g(\theta + c(\text{mod 1})) = f(g(\theta)) + \varepsilon h(\theta). \] (2)

If \( f(x) \) is linear (i.e., \( b = 0 \)), Eq. (2) is solved to give

\[ g(\theta) = \varepsilon (\sin 2\pi (\theta - c) - a \sin 2\pi \theta) \]
\[ / ((1 - a \cos 2\pi c)^2 + a^2 \sin^2 2\pi c). \] (3)

When \( f(x) \) is nonlinear, it is difficult to obtain the analytic solution of Eq. (2). Thus we search for the solution of Eq. (2) numerically by iterating the functional mapping

\[ g_{n+1}(\theta + c(\text{mod 1})) = f(g_n(\theta)) + \varepsilon h(\theta). \] (4)

As a numerical technique, we replace \( c \) by a rational value \( c_n \) using a continued fraction expansion:

\[ c_n = 1/(n_1 + 1/n_2 + 1/n_3 + \cdots 1/n_k + \cdots). \] (5)

We study mainly the case \( c = (\sqrt{5} - 1)/2 \), when \( c_n \) is given by \( F_{n-1}/F_n \), where \( F_n \) is the Fibonacci sequence. Thus, functional map (4) is replaced by the \( F_n \)-dimensional mapping. The convergence of iteration (4) becomes slower and slower as \( \varepsilon \) approaches \( \varepsilon_c \), and no convergence is obtained for \( \varepsilon > \varepsilon_c \) within our numbers of iterations (5000).

The figures of the attractors tell us that the torus seems to be fractal as \( \varepsilon \approx \varepsilon_c \). To confirm this property, we measured the length of torus by changing the scales, i.e., we calculated the following quantities:

\[ L(j) = \frac{1}{j/2} \sum_{i=1}^{j/2} [(g(i + j)/F_n(\text{mod 1})) - g(i/F_n)]^2 + (j/F_n)^2]. \] (6)

If \( L(j) \approx j^{-\alpha} \) for small \( j \), the torus is fractal with the dimension \( d_f = 1 + \alpha \). The log-log plot of \( L(j) \) vs \( j \) is shown in Fig. 2, which was obtained...
from numerical iterations of map (4) with \( \varepsilon = 0.472 \sim \varepsilon_c \) and \( F_s = 28657 \) or 46368. As is seen from this figure, the torus at \( \varepsilon \sim \varepsilon_c \) is fractal with the fractal dimension \( d_f = 1.77 \pm 0.04 \).

We also counted the number of extremum points \( N_s \) by changing \( F_s \), by calculating the number of the integers \( j \) which satisfy \( |q((j+1)/F_s) - g(j/F_s)| \times |q(j/F_s) - g((j-1)/F_s)| < 0 \) \( (1 \leq j \leq F_s) \). For \( \varepsilon \ll \varepsilon_c \), the number \( N_s \) approaches a constant as \( F_s \) is increased, while it goes to infinity, showing the behavior \( N_s \propto F_s \) for \( \varepsilon \sim \varepsilon_c \). This result also confirms the fractal property of the torus at the onset of chaos (see Fig. 3).

As another example we consider the case \( c = \sqrt{2} \)

\[ f(x) = -x + x^3 \]

\[ h(\theta) = \sin(2\pi \theta) \]

The value of \( \varepsilon \) is 0.472 and \( F_s \) is 28657 (△) and 46368 (○). The function \( f(x) \) is given by \( f(x) = -x + x^3 \) and \( h(\theta) = \sin(2\pi \theta) \).

Fig. 2. Log-log plot of the length \( L(j) \). The value of \( \varepsilon \) is 0.472 and \( F_s \) is 28657 (△) and 46368 (○). The function \( f(x) \) is given by \( f(x) = -x + x^3 \) and \( h(\theta) = \sin(2\pi \theta) \).

Fig. 3. The number of extrema \( N_s \) as a function of \( F_s \) for \( f(x) = -x + x^3 \). The values of \( \varepsilon \) are 0.45 (○), 0.46 (△), 0.465 (□), 0.47 (○) and 0.472 (△).
where the continued fraction expansion is given by \( c = G_{k+1}/G_k \) \((G_{k+1} = 2G_k + G_{k-1})\). The log-log plot of \( L(j) \) is given in Fig. 4. As is seen from this figure, the torus becomes fractal at the onset of chaos, but its dimension \((1.85 \pm 0.03)\) differs from the case \( c = (\sqrt{5} - 1)/2 \).

To study the universality of fractalization of torus, we consider the case \( f(x) = x - 0.2 \times \sin(2ax) \). In this case also, fractalization phenomena occur as \( \varepsilon \) is increased. Log-log plot of \( L(j) \) is given in Fig. 5, where \( c = (\sqrt{5} - 1)/2 \) or \( \sqrt{2} - 1 \) and their continued fraction expansions have been used \((F_a = 46368 \text{ or } G_a = 33461)\). The dimensions seem to agree with the case \( f(x) = ax + bx^2 \) respectively, but we have to leave the detailed study on the universality as a future problem.

Fractalization of torus is also seen for the doubling of torus.\(^{10-15}\) The torus seems to be fractal when the doubling cascade of torus stops and chaos emerges. This problem will be studied by choosing \( f(x) = 1 - ax^2 \) as was shown in Ref. 15. Fractalization is also found in a problem of a three-torus,\(^{17,20}\) where a coupled circle map is investigated.

In this letter, we reported the discovery of fractalization of torus, which may give a new critical phenomenon at the transition from torus to chaos. Fractal torus was investigated as a basin boundary of the attraction for a complex mapping.\(^{20,21}\) In our case it appears as an attractor and will be more relevant to physical observations. It will be of importance to search for this phenomenon in differential equation systems (a system with incommensurate modulation will be a candidate) and also in experiments. Detailed results on critical phenomena with renormalization group analysis will be reported elsewhere.

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2) S. Ostlund, D. Rand, J. Sethna and E. D. Siggia, to appear in Physica D.
9) K. Kaneko, to be published.
12) M. Sano and Y. Sawada, in the same reference as 7).

22) J. P. Sethna and E. D. Siggia, preprint (submitted to *Physica D*.}

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**Note:** The page contains references to various scientific papers and preprints, indicating a discussion of topics in statistical mechanics and chaos theory. The text mentions authors like K. Kaneko, P. Berge, M. Sano, V. Franceschini, A. Arnone, P. H. Coullet, E. A. Spiegel, H. Daido, and others, referencing works from journals like *Physica Scripta* and *Physica D*. The references also point towards collaborative research in the field, with contributions from notable figures in the study of complex systems and fractal geometry.