Prog. Theor. Phys. Vol. 71, No. 5, May 1984, Progress Letters

Fractalization of Torus*)

Kunihiko KANEKO
Department of Physics, University of Tokyo, Tokyo 113
(Received February 18, 1984)

For a certain class of two-dimensional mappings, torus oscillates more and more strongly as the bifurcation parameter is increased. It becomes fractal at a critical point and chaos appears above the critical point. The discovery of this phenomenon, the calculation of fractal dimension, and the universality of the phenomenon are reported.

Various nonequilibrium systems show transition from torus to chaos. Its mechanism and critical phenomena have extensively been studied. Especially, the instability of phase motion has been analyzed in detail using a one-dimensional circle map.[1)~7)] The amplitude behavior of torus, however, has been less investigated.[8)~16)] In this letter, we study the oscillation of torus, which was observed in various two-dimensional mappings[8)~10)] and also in experiments.[11),12)] In usual two-dimensional mappings, however, the region of frequency lockings increases near the onset of chaos, which masks the oscillatory behavior.

To study the oscillatory behavior in more detail, we introduce here a model which has no locking by its construction. It is a modulation map given by

$$\begin{cases} x_{n+1}=f(x_n)+\varepsilon h(\theta_n), \\ \theta_{n+1}=\theta_n+c, \quad (\mathrm{mod}\ 1) \end{cases} \tag{1}$$

where c is fixed at an irrational number, thus, no locking appears and $h(\theta)$ is a periodic function of period 1. This type of mapping has also been studied in the doubling of torus[15)] and in the problem of three-torus.[17)] We take here $f(x)=ax+bx^2$ and $h(\theta)=\sin(2\pi\theta)$, and consider the case that there is a stable torus for small ε. For $\varepsilon\sim 0$, the torus is almost straight. As ε is increased, it oscillates more and more strongly till it collapses at some critical value $\varepsilon=\varepsilon_c$ and chaos emerges (see Fig. 1a), b) and c) for the attractors). The value ε_c is numerically confirmed by the calculations of Lyapunov exponents.

In order to investigate the oscillations of torus in more detail, we study the equation for the invariant curve. The invariant curve $x=g(\theta)$, if it exists, must obey the functional equation

$$g(\theta+c(\mathrm{mod}\ 1))=f(g(\theta))+\varepsilon h(\theta). \tag{2}$$

If $f(x)$ is linear (i.e., $b=0$), Eq. (2) is solved to give

$$g(\theta)=\varepsilon(\sin 2\pi(\theta-c)-a\sin 2\pi\theta) \\ /\{(1-a\cos 2\pi c)^2+a^2\sin^2 2\pi c\}. \tag{3}$$

When $f(x)$ is nonlinear, it is difficult to obtain the analytic solution of Eq. (2). Thus we search for the solution of Eq. (2) numerically by iterating the functional mapping

$$g_{n+1}(\theta+c(\mathrm{mod}\ 1))=f(g_n(\theta))+\varepsilon h(\theta). \tag{4}$$

As a numerical technique, we replace c by a rational value c_k using a continued fraction expansion[18)]

$$c_k=1/\{n_1+\{1/n_2+\{1/n_3+\cdots 1/n_k\}\cdots\}. \tag{5}$$

We study mainly the case $c=(\sqrt{5}-1)/2$, when c_k is given by F_{k-1}/F_k, where F_k is the Fibonacci sequence.[18)] Thus, functional map (4) is replaced by the F_k-dimensional mapping. The convergence of iteration (4) becomes slower and slower as ε approaches ε_c, and no convergence is obtained for $\varepsilon \gtrsim \varepsilon_c$ within our numbers of iterations (5000).

The figures of the attractors tell us that the torus seems to be fractal[19)] as $\varepsilon\sim\varepsilon_c$. To confirm this property, we measured the length of torus by changing the scales, i.e., we calculated the following quantities:

$$L(j)=\frac{1}{j}\sum_{i=1}^{F_k}[\{g((i+j)/F_k(\mathrm{mod}\ 1)) \\ -g(i/F_k)\}^2+(j/F_k)^2]^{1/2}. \tag{6}$$

If $L(j)\propto j^{-\alpha}$ for small j, the torus is fractal with the dimension $d_F=1+\alpha$.[19)] The log-log plot of $L(j)$ vs j is shown in Fig. 2, which was obtained

*) Some parts of this work were reported at the KSI Conference "Chaos and Statistical Mechanics" (Kyoto, 1983, September).

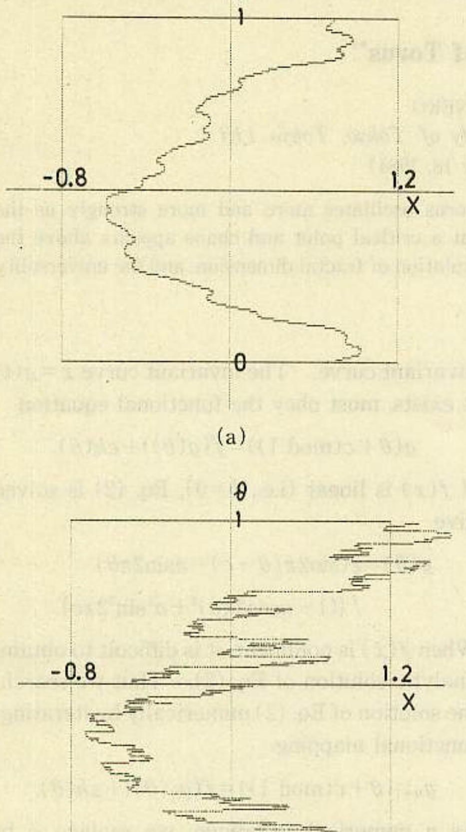

Fig. 1. Attractor of the map (1) with $f(x)=-x+x^2$ and $h(\theta)=\sin(2\pi\theta)$ and $c=(\sqrt{5}-1)/2$. The valuse of ε are 0.36 (torus) for (a), 0.46 (torus) for (b) and 0.49 (chaos) for (c).

from numerical iterations of map (4) with $\varepsilon=0.472\sim\varepsilon_c$ and $F_k=28657$ or 46368. As is seen from this figure, the torus at $\varepsilon\sim\varepsilon_c$ is fractal with the fractal dimension $d_F=1.77\pm0.04$.

We also counted the number of extremum points N_k by changing F_k, by calculating the number of the integers j which satisfy $\{g((j+1)/F_k)-g(j/F_k)\}\times\{g(j/F_k)-g((j-1)/F_k)\}<0$ $(1\leq j\leq F_k)$. For $\varepsilon\ll\varepsilon_c$, the number N_k approaches a constant as F_k is increased, while it goes to infinity, showing the behavior $N_k\propto F_k$ for $\varepsilon\sim\varepsilon_c$. This result also confirms the fractal property of the torus at the onset of chaos (see Fig. 3).

As another example we consider the case $c=\sqrt{2}$

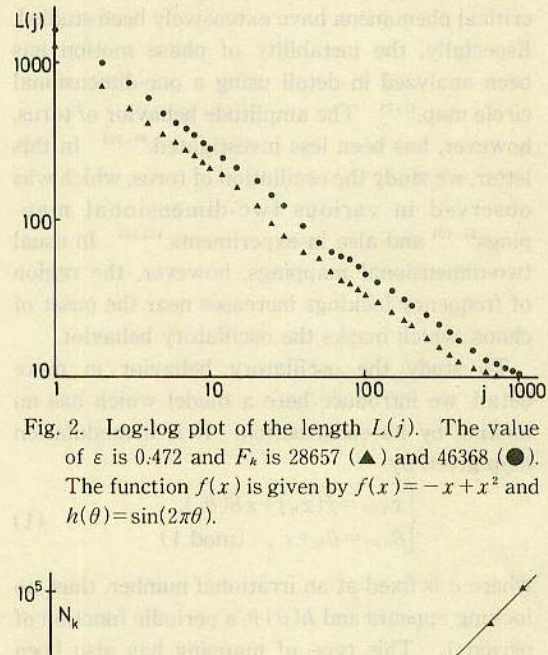

Fig. 2. Log-log plot of the length $L(j)$. The value of ε is 0.472 and F_k is 28657 (▲) and 46368 (●). The function $f(x)$ is given by $f(x)=-x+x^2$ and $h(\theta)=\sin(2\pi\theta)$.

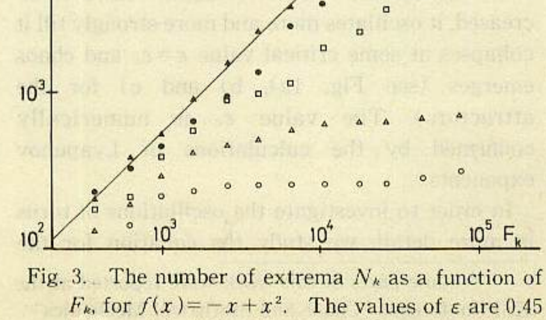

Fig. 3. The number of extrema N_k as a function of F_k, for $f(x)=-x+x^2$. The values of ε are 0.45 (○), 0.46 (△), 0.465 (□), 0.47 (●) and 0.472 (▲).

-1, where the continued fraction expansion is given by $c_k = G_{k-1}/G_k$ ($G_{k+1} = 2G_k + G_{k-1}$). The log-log plot of $L(j)$ is given in Fig. 4. As is seen from this figure, the torus becomes fractal at the onset of chaos, but its dimension (1.85 ± 0.03) differs from the case $c = (\sqrt{5}-1)/2$.

To study the universality of fractalization of torus, we consider the case $f(x) = x - 0.2 \times \sin(2\pi x)$. In this case also, fractalization phenomena occur as ε is increased. Log-log plot of $L(j)$ is given in Fig. 5, where c is $(\sqrt{5}-1)/2$ or $\sqrt{2}-1$ and their continued fraction expansions have been used ($F_k = 46368$ or $G_k = 33461$). The dimensions seem to agree with the case $f(x) = ax + bx^2$ respectively, but we have to leave the

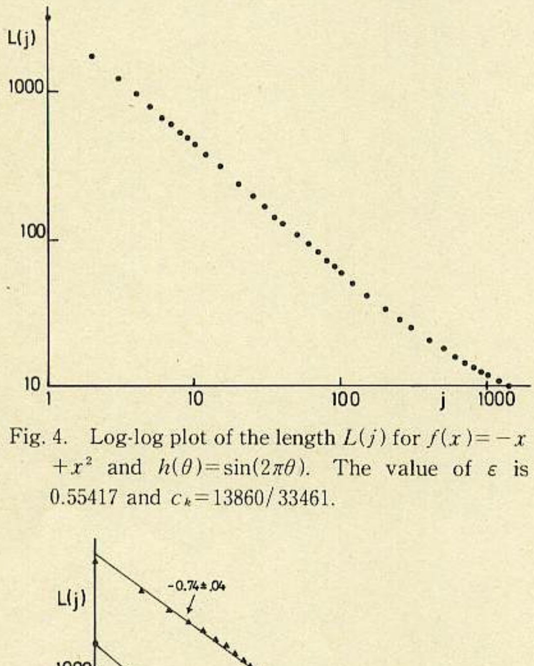

Fig. 4. Log-log plot of the length $L(j)$ for $f(x) = -x + x^2$ and $h(\theta) = \sin(2\pi\theta)$. The value of ε is 0.55417 and $c_k = 13860/33461$.

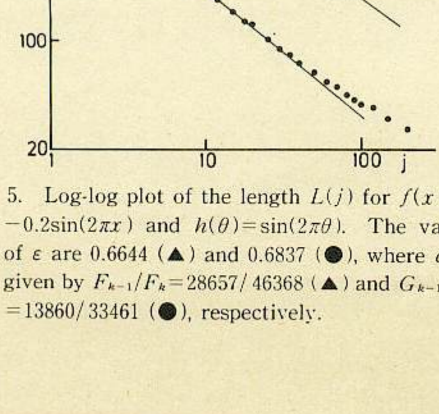

Fig. 5. Log-log plot of the length $L(j)$ for $f(x) = x - 0.2\sin(2\pi x)$ and $h(\theta) = \sin(2\pi\theta)$. The values of ε are 0.6644 (▲) and 0.6837 (●), where c_k is given by $F_{k-1}/F_k = 28657/46368$ (▲) and $G_{k-1}/G_k = 13860/33461$ (●), respectively.

detailed study on the universality as a future problem.

Fractalization of torus is also seen for the doubling of torus.[13)~15)] The torus seems to be fractal when the doubling cascade of torus stops and chaos emerges. This problem will be studied by choosing $f(x) = 1 - ax^2$ as was shown in Ref. 15). Fractalization is also found in a problem of a three-torus,[17),22)] where a coupled circle map is investigated.

In this letter, we reported the discovery of fractalization of torus, which may give a new critical phenomenon at the transition from torus to chaos. Fractal torus was investigated as a basin boundary of the attraction for a complex mapping.[20),21)] In our case it appears as an attractor and will be more relevant to physical observations. It will be of importance to search for this phenomenon in differential equation systems (a system with incommensurate modulation will be a candidate) and also in experiments. Detailed results on critical phenomena with renormalization group analysis will be reported elsewhere.

The author would like to thank Professor M. Suzuki for useful discussions and critical reading of the manuscript. He is also indebted to Professor Y. Takahashi for useful comments, to Dr. S. Thomae for pointing out Ref. 11), and to Dr. B. B. Mandelbrot and Dr. D. Rand for useful discussions. He would also like to thank LICEPP for the facilities of FACOM M190. This study was partially financed by the Scientific Research Fund of the Ministry of Education, Science and Culture.

1) S. J. Shenker, Physica **5D** (1982), 405.
M. J. Feigenbaum, L. P. Kadanoff and S. J. Shenker, ibid. 370.
2) S. Ostlund, D. Rand, J. Sethna and E. D. Siggia, to appear in Physica D.
3) K. Kaneko, Prog. Theor. Phys. **68** (1982), 669; **69** (1983), 403.
4) L. Glass and R. Perez, Phys. Rev. Lett. **48** (1982), 1772.
M. Schell, S. Fraser and R. Kapral, Phys. Rev. **A28** (1983), 1637.
5) L. P. Kadanoff, J. Stat. Phys. **31** (1983), 1.
6) M. H. Jensen, P. Bak and T. Bohr, Phys. Rev. Lett. **50** (1983), 1637.
7) K. Kaneko, to appear in *Turbulence and Chaotic Phenomena in Fluids* (North Holland, ed. T. Tatsumi).
8) K. Kaneko, Prog. Theor. Phys. **69** (1983), 1427.

9) K. Kaneko, to be published.
10) K. Kaneko, to appear in *Chaos and Statistical Mechanics* (Springer, ed. Y. Kuramoto).
11) P. Berge, to appear in Physica Scripta.
12) M. Sano and Y. Sawada, in the same reference as 7).
13) V. Franceschini, Physica **6D** (1983), 285 and preprint.
14) A. Arneodo, P. H. Coullet and E. A. Spiegel, Phys. Lett. **94A** (1983), 1.
15) K. Kaneko, Prog. Theor. Phys. **69** (1983), 1806.
16) H. Daido, to appear in Prog. Theor. Phys.
17) K. Kaneko, "Fates of Three-Torus I" to appear in Prog. Theor. Phys.
See also C. Grebogi, E. Ott and J. A. Yorke, Phys. Rev. Lett. **51** (1983), 339.
18) J. M. Greene, J. Math. Phys. **20** (1979), 1183. See also Refs. 1) and 2).
19) B. B. Mandelbrot, *The Fractral Geometry of Nature* (Freeman, San Francisco, 1982).
20) S. Manton and M. Nauenberg, Comm. Math. Phys. **89** (1983), 555.
21) M. Widom, preprint.
22) J. P. Sethna and E. D. Siggia, preprint (submitted to Physica D).