Progress of Theoretical Physics, Vol. 65, No. 3, March 1981

Phase Transition and Slowing Down in Non-Equilibrium Stochastic Processes

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(Received August 16, 1980)

A general criterion of appearance of slowing down in non-equilibrium stochastic processes is proposed. Many examples for this general criterion are shown, in which a phenomenon of slowing down occurs at a certain value of the relevant parameter. In particular, a generalized scaling treatment of transient phenomena is effectively applied to deriving the relaxation spectra of some multiplicative stochastic processes. The concept of asymptotic slowing down for finite systems is also proposed.

§ 1. Introduction

Recently many authors[1)~10)] are interested in non-equilibrium phase transition. However, nobody has yet established a general relation between a phase transition in non-equilibrium systems and a phenomenon of slowing down of relaxation. In this paper, we propose a general criterion of appearance of the slowing down to study the above problem.

The phase transition point γ_p in a non-equilibrium system is usually defined by the point of the relevant parameter γ contained in the system at which the profile of the stationary distribution function changes drastically, as is shown in Fig. 1. That is, γ_p is a bifurcation point[1)] in the parameter space of γ.

In the present paper, we consider the general nonlinear Langevin equation or stochastic differential equation

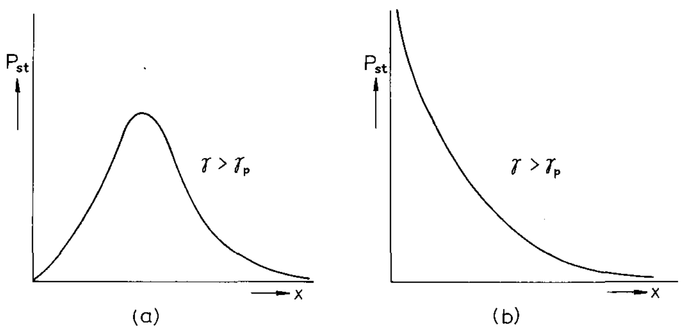

Fig. 1. The probability profile changes drastically at $\gamma = \gamma_p$.

$$\frac{d}{dt}x(t)=\alpha(x)+\beta(x)\eta(t). \tag{1·1}$$

Here $\eta(t)$ is a random force and $\alpha(x)$ contains γ. For simplicity we assume, except in § 7, that $\eta(t)$ is Gaussian and white, that is,

$$\langle\eta(t)\eta(t')\rangle=2\varepsilon\delta(t-t'), \tag{1·2}$$

where the parameter ε denotes the strength of the random force and it is a very important smallness parameter in our arguments. All the multiplicative noises in this paper, unless otherwise specified, are defined as the Stratonovich type. Now the corresponding Fokker-Planck equation takes the form

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}[\alpha(x)+\varepsilon\beta(x)\beta'(x)]P+\varepsilon\frac{\partial^2}{\partial x^2}\beta^2(x)P\,. \tag{1·3}$$

Then the stationary solution is given by

$$P_{\rm st}(x)=N_0\exp\left[\frac{1}{\varepsilon}\varphi(x,\gamma)\right];\quad \varphi(x,\gamma)=-\varepsilon\log\beta(x)+\int^x\frac{\alpha(y)}{\beta^2(y)}dy\,. \tag{1·4}$$

Here N_0 is the normalization constant. The phase transition point γ_p is easily obtained from (1·4). Namely, γ_p is determined by the condition that the most probable point x_0 begins to change from a stable state to an unstable state. This results in

$$\varphi'(x_0,\gamma_p)=[\alpha(x_0)-\varepsilon\beta(x_0)\beta'(x_0)]\beta^{-2}(x_0)=0 \tag{1·5}$$

and

$$\varphi''(x_0,\gamma_p)=[\alpha'(x_0)-\varepsilon\{[\beta'(x_0)]^2+\beta(x_0)\beta''(x_0)\}]\beta^{-2}(x_0)=0\,. \tag{1·6}$$

For example, in the non-multiplicative case $\beta(x)=1$, we obtain

$$\varphi'(x_0,\gamma_p)=\alpha(x_0)=0\quad\text{and}\quad\varphi''(x_0,\gamma_p)=\alpha'(x_0)=0\,. \tag{1·7}$$

That is, a phase transition occurs at the point at which the coefficient in the linear term in $\alpha(x)$ changes its sign, as is well-known.

Now our question is whether or not the slowing down occurs at this phase transition point. It is well-known[11)~13)] that there always occurs a critical slowing down at the phase transition point near the equilibrium state in the thermodynamic limit. It is, however, not necessarily so in general open systems. In fact, we can show in the present paper examples in which there is a phase transition without critical slowing down.

In § 2, we propose a general criterion of appearance of the slowing down and explain its validity intuitively as well as by using some typical examples. In § 3, a direct method is applied to obtain formal exact solutions in some specific models. The extended scaling method is shown in § 4 to be very powerful in obtaining the exact relaxation spectra of some multiplicative stochastic processes. To explain why the extended scaling treatment yields the exact relaxation spectra of some specific models, we study in § 5 the scaling property of solutions with the use of a linear scale transformation. In § 6, we introduce a concept of asymptotic slowing down. The scaling theory is also extended in § 7 to multiplicative stochastic processes with the two-level noise. Applications of the Itô type stochastic differential equations are shown in § 8. A discussion is given in § 9.

The present results were briefly reported in review articles[9),10)] by one of the present authors.

§ 2. General criterion of appearance of slowing down

We explain here a general mechanism of the slowing down of relaxation. By the slowing down, we mean here the divergence of the relaxation time, that is, $\tau \to \infty$ at a certain point $\gamma = \gamma_0$.

General criterion of appearance of the slowing down: For a slowing down to appear, at least one physical mode Q should exist such that

$$\langle Q \rangle_{st} = \pm\infty \qquad \text{at some point } \gamma_0 \,, \tag{2·1}$$

or the stationary distribution $P_{st}(x)$ or the initial distribution $P_{ini}(x)$ should become unnormalizable at some point γ_0.

Here, the stochastic variable $Q(x)$ itself is assumed not to contain a trivial singularity like $(\gamma - \gamma_0)^{-1}$ at $\gamma = \gamma_0$, because it is irrelevant to the slowing down.

The above statement on the slowing down is the summary of our present investigation on the critical slowing down in several systems. To explain this general criterion, we classify the mechanism of slowing down in the following.

(i) *The case in which the average $\langle Q \rangle_{st}$ in the stationary state diverges*

For simplicity, we assume here that the lowest non-zero eigenvalue of the Fokker-Planck operator in (1·3), λ_1, is isolated from other ones. Then, the average $\langle Q(t) \rangle$ is expected to have the following asymptotic behavior:

$$\langle Q(t) \rangle \sim A e^{-\lambda_1 t} + B \tag{2·2}$$

or

$$\frac{d}{dt}\langle Q(t) \rangle \sim -\lambda_1 \langle Q(t) \rangle + C; \quad C = \lambda_1 B \tag{2·3}$$

for large t. The stationary value $\langle Q \rangle_{st}$ is expressed as

$$\langle Q \rangle_{st} = \frac{C}{\lambda_1} \qquad \text{near} \quad \gamma = \gamma_0 \,, \tag{2·4}$$

if $C(\gamma=\gamma_0)\neq 0$. Thus, the divergence of $\langle Q\rangle_{st}$ is equivalent to $\lambda_1=0$, that is, the appearance of slowing down near $\gamma=\gamma_0$. Conversely, the latter assures that $\langle Q\rangle_{st}=\pm\infty$, if $C(\gamma=\gamma_0)\neq 0$. The condition that $C(\gamma=\gamma_0)\neq 0$ is the one for a phenomenon of slowing down to be classified into the present category (i).

(ii) *The case in which $P_{st}(x)$ becomes unnormalizable at $\gamma=\gamma_0$*

It is expected in this case that the approach of $P(x, t)$ to the stationary state $P_{st}(x)$ becomes very slow near $\gamma=\gamma_0$. That is, the eigenvalue spectra should contain a "critical" eigenvalue λ_1 which vanishes at $\gamma=\gamma_0$. This may be reflected in the relaxation of moments $\langle x^n(t)\rangle$, and consequently a phenomena of slowing down may occur at $\gamma=\gamma_0$. The simplest example of this case is the following linear stochastic process:

$$\frac{d}{dt}x(t)=-\gamma x+\eta(t). \tag{2·5}$$

The solution of (2·5) is given by

$$x(t)=x(0)e^{-\gamma t}+e^{-\gamma t}\int_0^t e^{\gamma s}\eta(s)ds\,, \tag{2·6}$$

as is well-known. The average $\langle x(t)\rangle$ shows the following slowing down:

$$\langle x(t)\rangle=\langle x(0)\rangle e^{-\gamma t} \tag{2·7}$$

near $\gamma=0$. The relaxation time τ is given by $\tau=1/\gamma$ and τ becomes infinite at $\gamma=0$. The stationary distribution function is also given by

$$P_{st}(x)=\frac{1}{\sqrt{2\pi\varepsilon\sigma_e}}\exp\left(-\frac{x^2}{2\varepsilon\sigma_e}\right), \tag{2·8}$$

where $\sigma_e=1/\gamma$. Therefore, P_{st} is unnormalizable for $\gamma\leq 0$. This feature is reflected in the appearance of slowing down of moments. It should be also remarked that the fluctuation $\langle x^2\rangle_{st}(\equiv\varepsilon\sigma_e)$ diverges at $\gamma=0$ in this linear system. Namely, the critical slowing down is associated with the divergence of the fluctuation in a linear system.

The following multiplicative stochastic process gives another simple example:

$$\frac{d}{dt}x(t)=-\gamma x+x\eta(t). \tag{2·9}$$

The solution of (2·9) is easily given by

$$x(t)=x(0)\exp\left(-\gamma t+\int_0^t\eta(s)ds\right). \tag{2·10}$$

Then, the average $\langle x^n(t)\rangle$ shows the following slowing down:

$$\langle x^n(t)\rangle=\langle x^n(0)\rangle e^{-n(\gamma-n\varepsilon)t} \tag{2·11}$$

near $\gamma=n\varepsilon$. The stationary distribution function is expressed as[5]

$$P_{st}(x)=N_0 x^{-(\gamma+\varepsilon)/\varepsilon}. \tag{2·12}$$

This is always unnormalizable. This point $\gamma=n\varepsilon$ of the slowing down corresponds to the divergent behavior of the integral of $x^n P_{st}(x)$ for large x.

(iii) *The case in which $P_{ini}(x)$ is "singular", namely unnormalizable at some point $\gamma=\gamma_0$*

It is expected in this case that the approach of the singular (or "almost divergent" near $\gamma=\gamma_0$) $P_{ini}(x)$ to the finite stationary state becomes very slow near $\gamma=\gamma_0$. An example of this case will be given later in the Schenzle-Brand model[5] near $\gamma=\varepsilon$.

It seems rather difficult to prove the above criterion rigorously. Here, we demonstrate how the above general criterion works for typical models.

(A) *The Schenzle-Brand model* We consider the following multiplicative stochastic processes:[5),9),10)]

$$\frac{d}{dt}x(t)=\gamma x-gx^m+x\eta(t);\quad g>0. \tag{2·13}$$

Then it is quite easy[9),10)] to show that the quantity $Q^{(SB)}\equiv x^{1-m}$ is the relevant divergent mode, that is,

$$\langle Q^{(SB)}\rangle_{st}=\frac{g}{\gamma-\gamma_0}\to\infty \quad \text{at} \quad \gamma=\gamma_0+0, \tag{2·14}$$

because $P_{st}(x)$ is given by[5]

$$P_{st}(x)=N_0 x^{\frac{\gamma-\varepsilon}{\varepsilon}}e^{-gx^{m-1}/\varepsilon(m-1)}, \tag{2·15}$$

where $\gamma_0=(m-1)\varepsilon$. The above general criterion states that there is a slowing down associated with this quantity $Q^{(SB)}$ and that the relaxation time τ should be proportional to $(\gamma-\gamma_0)^{-1}$. In fact, it is shown[9),10)] rigorously that

$$\langle Q^{SB}\rangle_t=\left(\langle Q^{(SB)}\rangle_0-\frac{g}{\gamma-\gamma_0}\right)e^{-(m-1)(\gamma-\gamma_0)t}+\frac{g}{\gamma-\gamma_0}. \tag{2·16}$$

Thus, there exists, at least, one "critical" mode with slowing down in this model. However, Schenzle and Brand[5] overlooked this fact and they concluded that there is no "critical" slowing down at all.

(B) *A new model (to be referred to as SKS-model)* The following model can be solved formally:

$$\frac{d}{dt}x(t)=\gamma x-gx^m+x^m\eta(t);\quad g>0. \tag{2·17}$$

This can be derived, for example, as a model of an autocatalytic reaction, i.e., A

$+X\rightleftarrows B+mX$. If B is fluctuating around the average, then we obtain the model (2·17). Alternatively, this is also interpreted as a stochastic model of the nonlinear system $\dot{x}=\gamma x-g(t)x^m$ with a fluctuating nonlinear coupling $g(t)\equiv g-\eta(t)$. For this model, the quantity $Q^{(\mathrm{SKS})}\equiv x^{1-m}$ is again the relevant divergent mode, that is,

$$\langle Q^{(\mathrm{SKS})}\rangle_{\mathrm{st}}=\frac{g}{\gamma-\gamma_0}\to\infty \quad \text{at} \quad \gamma=\gamma_0+0\,, \tag{2·18}$$

where $\gamma_0\equiv 0$. Thus, the relaxation time τ should be proportional to γ^{-1}. In fact, we can show rigorously again that

$$\langle Q^{(\mathrm{SKS})}\rangle_t=\left(\langle Q^{(\mathrm{SKS})}\rangle_0-\frac{g}{\gamma}\right)e^{-(m-1)\gamma t}+\frac{g}{\gamma}\,. \tag{2·19}$$

That is, $\langle x(t)^{1-m}\rangle$ shows a slowing down at $\gamma=0$ irrespectively of the value of the strength ε of the Gàussian white random force $\eta(t)$.

(C) *Arnold-Horsthemke-Lefever model (AHL-model)* Arnold et al.[14] proposed the following stochastic system whose phase transition is induced purely by an external noise:

$$\frac{d}{dt}x(t)=\frac{1}{2}-x+x(1-x)(\beta+\eta(t)), \tag{2·20}$$

where β is a constant. The stationary distribution function is given by

$$P_{\mathrm{st}}(x)=\frac{N_0}{x(1-x)}\exp\left[-\frac{1}{2\varepsilon x(1-x)}-\frac{\beta}{\varepsilon}\log\left(\frac{1-x}{x}\right)\right]. \tag{2·21}$$

The average $\langle x\rangle_{\mathrm{st}}$ is always finite in this system and $P_{\mathrm{st}}(x)$ is normalizable for finite ε. Thus, from our general criterion, we may conclude that $\langle x(t)\rangle$ shows no slowing down in the AHL-model. Kabashima et al.[15] performed an experiment corresponding to this model by using electric circuits, and they found no slowing down. Our above arguments explain very well this experimental result.

(D) *Non-multiplicative stochastic processes* We consider here the following non-multiplicative process:

$$\frac{d}{dt}x(t)=a(x)+\eta(t). \tag{2·22}$$

It is well-known[16),17)] that no fluctuation diverges in the stationary state for $\varepsilon\neq 0$. Therefore our general criterion concludes that there is no slowing down for $\varepsilon\neq 0$ in the non-multiplicative stochastic process (2·22). Only in the limit of small ε, the fluctuation diverges at the critical point and consequently the critical slowing down occurs in this limit. Van Hove's theory corresponds to the case $\varepsilon=0$, that is, to the deterministic system. The critical slowing down is predicted in this theory, as is well-known:

$$\frac{\partial}{\partial t}M(t)=-\Gamma\frac{\partial f}{\partial M};\quad f=f_0+\frac{1}{2}\chi^{-1}M^2\cdots. \tag{2·23}$$

Here, M denotes the order parameter. Equation (2·23) yields

$$\frac{\partial}{\partial t}M(t)=-(\Gamma/\chi)M(t). \tag{2·24}$$

The solution shows the following exponential decay:

$$M(t)=M(0)\exp(-t/\tau) \tag{2·25}$$

with the relaxation time

$$\tau=\chi/\Gamma\sim(T-T_c)^{-\gamma}\to\infty \quad \text{at} \quad T=T_c\,. \tag{2·26}$$

The fluctuation $\langle M^2\rangle_{st}$ diverges at $T=T_c$ in this system and this corresponds to the case (ii) in the above criterion.

As is easily seen from the above illustrations, the slowing down is always associated with the instability of a certain kind of physical quantity.

§ 3. Direct method—formal exact solutions

In this section, we investigate some exactly soluble models to find many kinds of slowing down.

It will be instructive to classify the stochastic processes (1·1) into exactly soluble systems and others. Here we restrict ourselves to a special method for it. That is, we are interested in the nonlinear transformation $\xi=f(x)$ which transforms (1·1) into a linear stochastic process of the form

$$\frac{d}{dt}\xi(t)=(a+b\eta(t))\xi+c+d\eta(t). \tag{3·1}$$

We can prove easily the following statement.

Theorem of Solvability: *If* $\alpha(x)$ *and* $\beta(x)$ *in* (1·1) *satisfy the condition*

$$\frac{d}{dx}\left(\frac{\alpha(x)}{\beta(x)}\right)=\frac{-1}{\beta(x)}\left\{b\left(\frac{\alpha(x)}{\beta(x)}\right)-a\right\} \tag{3·2}$$

for appropriate constants a *and* b, *then* (1·1) *can be transformed into a linear differential equation and consequently it can be solved formally.*

For the case $b=0$, see a paper by Sancho and San Miguel.[23] There are several useful examples of exactly soluble models in the above sense.

(A) *Formal solution of Schenzle-Brand model* The model (2·13) is the most interesting example of the above theorem. With the use of the nonlinear transformation $\xi=x^{1-m}$, Eq. (2·13) is transformed into the linear differential equation

$$\frac{d\xi}{dt}=(1-m)(\gamma+\eta(t))\xi+(m-1)g\,. \tag{3·3}$$

The solution of (2·13) is given by

$$x(t)=\exp\left[\gamma t+\int_0^t \eta(t')dt'\right]$$
$$\times\left[g(m-1)\int_0^t \exp\left\{(m-1)\left(\gamma t'+\int_0^{t'}\eta(s)ds\right)\right\}dt'+x(0)^{1-m}\right]^{1/(1-m)}. \tag{3·4}$$

This yields the result (2·16) immediately. It is, however, rather difficult to calculate explicitly general fluctuations such as $\langle x^2(t)\rangle$ from the above formal solution (3·4), because we must take the average over the random force $\eta(t)$ in the denominator of (3·4). An explicit expression for the first moment $\langle x(t)\rangle$ will be given on the basis of the scaling theory in the next section, to obtain the relaxation spectra of this system.

(B) *SKS-model* The second example of exactly soluble models is described by (2·17). This is also transformed into the linear equation

$$\frac{d\xi}{dt}=(1-m)\gamma\xi+(m-1)(g-\eta(t)), \tag{3·5}$$

in terms of the same nonlinear transformation $\xi=x^{1-m}$. Consequently the formal solution of (2·17) is given by

$$x(t)=e^{\gamma t}\left[(m-1)\int_0^t e^{(m-1)\gamma t'}(g-\eta(t'))dt'+x(0)^{1-m}\right]^{1/(1-m)}. \tag{3·6}$$

This gives the result (2·19). The first moment $\langle x(t)\rangle$ is easily given in the form

$$\langle x(t)\rangle=\sum_{n=0}^{\infty}a_n e^{-\lambda_n t}\,, \tag{3·7}$$

where $\{a_n\}$ are functions of g, ε, γ, m and $x(0)$, and the spectral eigenvalue λ_n is given by

$$\lambda_n=n(m-1)\gamma\,. \tag{3·8}$$

This result is also obatined easily by solving the Schrödinger-type equation corresponding to (2·17). For details, see Appendix A. The stationary solution of this model is given by

$$P_{st}(x)=N_0x^{-m}\exp\left[\frac{1}{\varepsilon}\left\{-\frac{\gamma}{2(m-1)}x^{2(1-m)}+\frac{g}{m-1}x^{1-m}\right\}\right], \tag{3·9}$$

where N_0 is the normalization factor. Therefore, $P_{st}(x)$ changes drastically at $\gamma=0$, as shown in Fig. 1. It should be noted, however, that P_{st} is not normalizable for $\gamma<0$. The physical region is restricted to $\gamma>0$. Thus, the "phase transition point" $\gamma=0$ is different from the ordinary ones, and it is rather artificial. The slowing down in this system is classified in the case (ii) of the criterion.

There are several other models that can be transformed into exactly soluble Schrödinger-type equations. We consider the following stochastic processes:

$$\frac{d}{dt}x(t)=\gamma x^l-gx^m+x^n\eta(t). \qquad (3\cdot10)$$

For exactly soluble sets of l, m and n, see Table I.

Table I. Spectra for $\dot{x}=\gamma x^l-gx^m+x^n\eta(t)$, where $V(z)$ denotes potentials in the corresponding Schrödinger-type equations.

type	l	m	n	potential $V(z)$	spectra λ_k
I	1	m	m	$Az^2/2$	$k\gamma(m-1)$
II	1	m	1	$e^{-2az}-2e^{-az}$	$k(m-1)(\gamma-k(m-1)\varepsilon)$ and continuum
III	1	$2n-1$	n	A/z^2+Bz^2	$2k\gamma(n-1)$
IV	n	$2n-1$	n	$A/z+B/z^2$	$(\gamma^2/4\varepsilon)\{1-(1+2\varepsilon k(n-1)/g)^{-2}\}$ and continuum

§ 4. Scaling treatment for multiplicative stochastic processes

We apply here the extended scaling method[9),10)] to the multiplicative stochastic process (1·1) to obtain the relaxation spectra. According to the general procedure presented in the previous papers,[9),10),19)] we make use of the following nonlinear transformation:

$$\xi(x,t)=F^{-1}(e^{-\gamma t}F(x)), \qquad (4\cdot1)$$

where $\gamma=a'(0)>0$ and

$$F(x)\equiv\exp\int_{a_0}^{x}\frac{\gamma}{a(y)}dy=x+\cdots \qquad (4\cdot2)$$

with a_0 to be determined so that $F'(0)=1$. Then, we obtain the transformed equation

$$\frac{d}{dt}\xi=G(\xi,t)\eta(t), \qquad (4\cdot3)$$

where

$$G(\xi,t)\equiv e^{-\gamma t}F'(F^{-1}(e^{\gamma t}F(\xi)))\beta(F^{-1}(e^{\gamma t}F(\xi)))/F'(\xi). \qquad (4\cdot4)$$

If we assume that

$$\beta(x)=x^{p}+O(x^{p+\delta});\quad \delta>0 \tag{4·5}$$

near $x=0$ (unstable point), for an arbitrary positive number p, then we can show[20),21)] that $G(\xi, t)$ in (4·4) is replaced "asymptotically" by

$$G_{\rm sc}(\xi,\ t)=e^{(p-1)\gamma t}\xi^{p}\,. \tag{4·6}$$

By "asymptotically", we mean[9),10),20),21)] that $\varepsilon\to 0$ and $t\to\infty$ with $\varepsilon\exp(2\gamma t)$ being fixed. We call this the scaling limit. Thus, the original multiplicative stochastic process (1·1) can be approximated by

$$\frac{d}{dt}\xi_{\rm sc}(t)=e^{(p-1)\gamma t}\xi_{\rm sc}^{p}(t)\eta(t), \tag{4·7}$$

in the scaling limit. The solution of this equation is given by

$$\xi_{\rm sc}(t)=x(0)\exp\left[\int_0^t\eta(s)ds\right]\qquad \text{for } p=1\,, \tag{4·8}$$

$$\xi_{\rm sc}(t)=\left[(1-p)\int_0^t e^{(p-1)\gamma s}\eta(s)ds+x(0)^{1-p}\right]^{1/(1-p)}\qquad \text{for } p\neq 1\,. \tag{4·9}$$

Consequently the scaling solution or renormalized solution[10)] is expressed by

$$x_{\rm sc}(t)=F^{-1}(e^{\gamma t}F(\xi_{\rm sc}(t))). \tag{4·10}$$

In general, the average of any quantity $Q(x)$ is expressed in this scaling theory as

$$\langle Q(x)\rangle_{\rm sc}=\langle Q(F^{-1}(e^{\gamma t}F(\xi_{\rm sc}(t))))\rangle\,. \tag{4·11}$$

The average over the random force $\eta(t)$ in (4·11) is easily taken as in Ref. 9). That is, for a fixed value of x_0, we expand $Q(x)$ in a power series of $\tilde{\eta}\equiv\int\{\exp[(p-1)\gamma s]\}\eta(s)ds$ for $p\neq 1$ and we use the Wick theorem to take the average of the product $\eta(t_1)\eta(t_2)\cdots\eta(t_{2m})$. For this purpose, we consider an arbitrary function $f(\tilde{\eta})$ and make a Taylor expansion as

$$\begin{aligned}\langle f(\tilde{\eta})\rangle&=\sum_{n=0}^{\infty}a_n\langle\tilde{\eta}^n\rangle\\&=\sum_{n=0}^{\infty}a_{2n}(2n-1)!!\langle\tilde{\eta}^2\rangle^n\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}\sum_{n=0}^{\infty}a_{2n}(\xi^2\langle\tilde{\eta}^2\rangle)^n\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}f(\xi\langle\tilde{\eta}^2\rangle^{1/2}).\end{aligned} \tag{4·12}$$

Here, we have

$$\langle\tilde{\eta}^2\rangle=\int_0^t ds\int_0^t ds' e^{(p-1)\gamma(s+s')}\langle\eta(s)\eta(s')\rangle$$

$$=\frac{\varepsilon}{(p-1)\gamma}(e^{2(p-1)\gamma t}-1) \tag{4·13}$$

for $p\neq 1$, and we have

$$\langle\tilde{\eta}^2\rangle=\lim_{p\to 1}\langle\tilde{\eta}^2\rangle(p)=2\varepsilon t \quad \text{for } p=1\,. \tag{4·14}$$

Thus, we arrive at the following results:

$$\langle Q(x)\rangle_{sc}(x(0))$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}Q(F^{-1}(e^{\gamma t}F([x(0)^{1-p}+(1-p)\xi\langle\tilde{\eta}^2\rangle^{1/2}]^{1/(1-p)}))) \tag{4·15}$$

for $p\neq 1$, and

$$\langle Q(x)\rangle_{sc}(x(0))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}Q(F^{-1}(e^{\gamma t}F(x(0)e^{\sqrt{2\varepsilon t}\,\xi}))) \tag{4·16}$$

for $p=1$. Equation (4·16) is also obtained by taking the limit $p\to 1$ in (4·15).

If we take the average over the initial value $x(0)$, then we obtain

$$\langle Q(x)\rangle_{sc}=\int_{-\infty}^{\infty}\langle Q(x)\rangle_{sc}(x(0))P_0(x(0))dx(0). \tag{4·17}$$

Furthermore if the distribution P_0 is Gaussian with $\langle x(0)\rangle=0$, then $\langle Q(x)\rangle_{sc}$ is reduced to

$$\langle Q(x)\rangle_{sc}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi$$
$$\times e^{-\xi^2/2}Q\left(F^{-1}\left(e^{\gamma t}F\left(\xi\left\{\frac{\varepsilon}{\gamma}(1-e^{-2\gamma t})+\langle x^2(0)\rangle\right\}^{1/2}\right)\right)\right) \tag{4·18}$$

for $p=0$. This is the result obtained in the previous paper.[9)]

For the Schenzle-Brand model, we obtain

$$x_{sc}(t)=\xi_{sc}(t)e^{\gamma t}\{1+ga(t)\xi_{sc}^{m-1}(t)\}^{1/(1-m)}\,, \tag{4·19}$$

where $\xi_{sc}(t)$ is given by (4·8) and

$$a(t)=\frac{1}{\gamma}\{\exp[\gamma(m-1)t]-1\}. \tag{4·20}$$

The average of $x(t)$ is given by

$$\langle x(t)\rangle_{sc}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}x(0)$$

$$\times e^{\sqrt{2\varepsilon t}\,\xi+\gamma t}\{1+ga(t)x(0)^{m-1}e^{(m-1)\sqrt{2\varepsilon t}\,\xi}\}^{1/(1-m)}\,. \tag{4·21}$$

As we are interested in the relaxation spectra and consequently in the long-time behavior of the system, we calculate the average $\langle x(t)\rangle$ explicitly in the form of the strong-coupling expansion as

$$\langle x(t)\rangle_{\mathrm{sc}}=\left(\frac{\gamma}{g}\right)^{1/(m-1)}\sum_{n=0}^{n_0}\left(\frac{\gamma}{gx(0)^{m-1}}\right)^n \times\begin{pmatrix}\frac{1}{1-m}\\ n\end{pmatrix}(1-e^{-\gamma(m-1)t})^{-n+1/(1-m)}e^{-\lambda_n t}+R(t). \tag{4·22}$$

Here, $R(t)$ is defined by

$$R(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\xi e^{-\xi^2/2}\left(\frac{\gamma}{g}\right)^{1/(m-1)} \times\begin{pmatrix}\frac{1}{1-m}\\ n\end{pmatrix}(1-e^{-\gamma(m-1)t})^{-n_0-m/(m-1)}(1+\theta h(\xi,t))^{-n_0-m/(m-1)}h(\xi,t)^{n_0+1}\,, \tag{4·23}$$

$$h(\xi,t)=\frac{1}{gx(0)^{m-1}a(t)}\exp[-(m-1)\sqrt{2\varepsilon t}\,\xi] \tag{4·24}$$

and

$$\lambda_n=n(m-1)[\gamma-n(m-1)\varepsilon] \tag{4·25}$$

for $n=1, 2, \cdots, n_0$, where $0<\theta<1$. Here the parameter n_0 should be determined so that the separation (4·22) may be meaningful. That is, $R(t)$ should be of order higher than the first term. As $R(t)$ is easily shown to be of the order of $\exp(-\lambda_{n_0+1}t)$ or of order higher than it for large t, we obtain the condition that $\lambda_{n_0}<\lambda_{n_0+1}$, namely

$$n_0<[\gamma-(m-1)\varepsilon]/[2(m-1)\varepsilon]. \tag{4·26}$$

Thus, n_0 is given by the maximum integer less than $[\gamma-(m-1)\varepsilon]/[2(m-1)\varepsilon]$. If we are interested in the eigenvalue λ_1, we obtain the condition $n_0\geq 1$. That is, we have

$$\gamma>3(m-1)\varepsilon\,. \tag{4·27}$$

This is a sufficient condition stronger than that obtained by Schenzle and Brand[5] using the Schrödinger-type equation. The weakest condition may be the vanishing of $R(t)$ for $t\to\infty$, that is, $\lambda_{n_0+1}>0$. From this condition, we obtain $(n_0'+1)(m-1)\varepsilon<\gamma$. If we are interested in λ_1, we get the restriction $\gamma>2(m-1)\varepsilon$, by putting $n_0'=1$ in the above condition. This happens to agree with that obtained by Schenzle and Brand. However, judging from our scaling treatment, it seems safer to take the former stronger sufficient condition in our approximation. In any case, the average $\langle x(t)\rangle$ shows no critical slowing down at $\gamma=(m-1)\varepsilon$, but there occurs seemingly a slowing down in the region $\gamma>2(m-1)\varepsilon$, which was obtained first by Schenzle and Brand.

It should be noted, however, that there exists a critical mode $x(t)^{1-m}$ which shows a slowing down at $\gamma=(m-1)\varepsilon$, as was shown in § 3. Thus all the eigenvalues (4·25) satisfying $\gamma>n(m-1)\varepsilon$ are necessary to describe various situations mentioned above. The above results agree essentially with those obtained by Schenzle and Brand.[5] Their discrete eigenvalues are, however, restricted to $n=1, 2, \cdots, n_0'$ (which is the maximum integer value less than $\gamma/[2\varepsilon(m-1)]$), because they have imposed the normalization condition on the wave function of the Schrödinger-type equation. The correct normalization condition should be imposed on the probability function of the Fokker-Planck equation. For more details, see Appendix B.

For the SKS-model described by (2·17), we obtain similarly the eigenvalue spectra

$$\lambda_n=n(m-1)\gamma\,, \qquad n=1, 2, \cdots . \tag{4·28}$$

The spectra of an exactly soluble model described by

$$\frac{d}{dt}x(t)=\gamma x-gx^{2m-1}+x^m\eta(t) \tag{4·29}$$

are also obtained by the scaling treatment and they are given by

$$\lambda_n=2n(m-1)\gamma, \qquad n=1, 2, \cdots . \tag{4·30}$$

The derivation of this result is given in Appendix C.

§ 5. Linear scaling transformation and scaling property of solutions

In order to understand why the exact discrete spectra have been obtained in § 4, we study here the scaling property of the spectra by introducing the following linear scaling transformation;

$$g'=\varepsilon^\alpha g, \quad t'=\varepsilon^\beta t, \quad x'=\varepsilon^y x, \quad \gamma'=\varepsilon^z\gamma \tag{5·1}$$

in the nonlinear Langevin equation

$$\frac{d}{dt}x(t)=\gamma x^l-gx^m+x^n\eta(t). \tag{5·2}$$

It is easily shown that (5·2) is invariant for the following choice of the exponents

$$\beta=\frac{m-1+2\alpha(n-1)}{m-2n+1},\quad y=\frac{-(\alpha+1)}{m-n+1},\quad z=\frac{(l-2n+1)\alpha-(m-l)}{m-2n+1}. \tag{5·3}$$

Therefore, the eigenspectra λ_k should have the following scaling form:

$$\lambda_k=\varepsilon^{(m-1+2\alpha(n-1))/(m-2n+1)}\hat{\lambda}_k(\gamma\varepsilon^{((l-2n+1)\alpha-(m-l))/(m-2n+1)},\ \varepsilon^\alpha g). \tag{5·4}$$

This is a generalization of the case that $n=0$ and $\gamma=0$ discussed by Kubo et al.[22) This scaling property leads to the following arguments:

(a) If $n=1$ and $l=2n-1=1$ (for the Schenzle-Brand model), then we have

$$\lambda_k=\varepsilon\hat{\lambda}_k(\gamma/\varepsilon,\ \varepsilon^\alpha g). \tag{5·5}$$

This means that the eigenvalues λ_k have the above scaling property (5·5) for an arbitrary value of α and consequently that $\{\lambda_k\}$ do not depend on g, that is, we have $\lambda_k=\varepsilon\hat{\lambda}_k(\gamma/\varepsilon)$. Therefore, the strong coupling expansion or $(1/g)$-expansion can give the exact spectra. Furthermore, if we assume that λ_k is a linear function of γ, then λ_k should be a linear function of ε, and consequently the scaling treatment which is correct up to the order of ε gives the exact result, as is shown in § 4.

(b) For the SKS-model ($l=1$, $m=n$), we have $\lambda_k=\varepsilon^{-2\alpha-1}\hat{\lambda}_k(\gamma\varepsilon^{2\alpha+1},\ \varepsilon^\alpha g)$ for any value of α. Consequently, λ_k should be scaled in the form

$$\lambda_k=\frac{g^2}{\varepsilon}f_k^{(\mathrm{SKS})}\left(\frac{\gamma\varepsilon}{g^2}\right). \tag{5·6}$$

If $\{\lambda_k\}$ do not depend on g, they do not depend on ε, either and they are a linear function of γ without a constant term. Conversely, if $\{\lambda_k\}$ are proportional to γ, as it is, then they do not depend on g and ε. This justifies the scaling derivation of (4·28).

(c) If $m=2n-1$, then only the case $\alpha=-1$ gives the correct scaling property

$$\lambda_k=\varepsilon\hat{\lambda}_k(\gamma/\varepsilon,\ g/\varepsilon). \tag{5·7}$$

Clearly the specific results for $l=n$ and $l=1$ shown in Table I satisfy this linear scaling property. It is quite interesting that $\{\lambda_k\}$ do not depend on g and ε in the case $l=1$ but that they depend on g in the case $l=n$ and consequently they cannot be obtained by the $(1/g)$-expansion for this case.

(d) In this way, it is possible, in general, to argue in what models the spectra do not depend on the nonlinearity g, and consequently to what models the $(1/g)$-expansion can be applied to obtain the spectra.

§ 6. Asymptotic critical slowing down

As we have discussed in § 2, no critical slowing down occurs in the non-multiplicative stochastic process (2·13) for a non-vanishing value of ε. We are here interested in the asymptotic behavior of the relaxation time or spectra for small ε. The minimum eigenvalue λ_1 takes the scaling form

$$\lambda_1=\varepsilon^{(m-1)/(m+1)}\hat{\lambda}(\gamma\varepsilon^{-(m-1)/(m+1)})=\gamma\tilde{\lambda}(\gamma\varepsilon^{-(m-1)/(m+1)}) \tag{6·1}$$

from Eq. (5·4) with $\alpha=0$. The relaxation time τ is given by $\tau=\lambda_1^{-1}$. Thus, τ becomes very large asymptotically as $\tau\propto\varepsilon^{-(m-1)/(m+1)}$ for small ε at the critical point $\gamma=0$. This may be called "asymptotic critical slowing down". This situation has been also discussed by Dekker and van Kampen[17] on the basis of numerical calculation.

Experimentally this asymptotic critical slowing down should be observed in ordinary situations, because the case $\varepsilon=0$ is realized only as the mathematical limit and because ε is small but non-zero in experimental situations.

§ 7. Slowing down in two-level noise systems

The extended scaling treatment in § 4 can be applied easily to the following two-level noise systems:

$$\frac{d}{dt}x(t)=(\gamma+I(t))x-gx^m\,, \tag{7·1}$$

where $I(t)$ is a two-level $(\pm\Delta)$ noise[7),8)] satisfying the relation

$$\langle I(t)I(t')\rangle=\Delta^2e^{-\lambda|t-t'|}\,. \tag{7·2}$$

The case $m=2$ has been studied in detail by Kitahara et al.[7),8)] in a different method.

Using the nonlinear transformation $x\to\xi_{\rm sc}$ as is described in § 4, we get

$$\langle\xi_{\rm sc}^{\alpha}(t)\rangle=x^{\alpha}(0)\left\langle\exp\left[\alpha\int_0^t I(s)ds\right]\right\rangle\equiv x^{\alpha}(0)f_{\alpha}(t). \tag{7·3}$$

Noting that, for two-level noise

$$f_{\alpha}(t)=1+\alpha^2\int_0^t ds_1\int_0^t ds_2\langle I(s_1)I(s_2)\rangle f_{\alpha}(s_2) \tag{7·4}$$

and using (7·2), we have

$$\frac{df_{\alpha}}{dt}=(\alpha\Delta)^2\int_0^t dse^{-\lambda(t-s)}f_{\alpha}(s). \tag{7·5}$$

The above equation is easily solved by Laplace transformation, and this results in

$$f_{\alpha}(t)=\frac{\mu_++\lambda}{\mu_+-\mu_-}e^{\mu_+t}+\frac{\mu_-+\lambda}{\mu_--\mu_+}e^{\mu_-t}\,, \tag{7·6}$$

where

$$\mu_{\pm}=\frac{1}{2}\{-\lambda\pm\sqrt{\lambda^2+4\Delta^2\alpha^2}\}. \tag{7·7}$$

From this, we obtain

$$\langle x(t)\rangle_{\rm sc}=\left(\frac{\gamma}{g}\right)^{1/(m-1)}(1-e^{-(m-1)\gamma t})^{1/(1-m)}$$
$$\times\sum_{n=0}\left(\frac{\gamma}{g}\right)^n\binom{\frac{1}{1-m}}{n}\frac{e^{-(m-1)n\gamma t}f_{(1-m)n}(t)}{(1-e^{-(m-1)\gamma t})^n}. \tag{7·8}$$

Making such considerations as are given in § 4, we obtain the following discrete spectra:

$$\lambda_n=\frac{1}{2}[\lambda+2n(m-1)\gamma-\sqrt{\lambda^2+4n^2(m-1)^2\Delta^2}] \tag{7·9}$$

for $n=1, 2, \cdots, n_0$, where n_0 denotes the maximum integer that satisfies $\lambda_{n_0}<\lambda_{n0+1}$.

§ 8. Applications of the Itô type differential equations

We have used the Stratonovich-type stochastic differential equations up to §7. However, it is sometimes useful to make use of the Itô-type stochastic differential equations

$$dx=f(x, t)dt+g(x, t)dw \qquad \text{(Itô)} \tag{8·1}$$

with $(dw)^2=2\varepsilon dt$, $dw\cdot dt=0$ and $(dt)^2=0$. The well-known transformation formula yields

$$dx=\alpha(x, t)dt+\beta(x, t)dw \qquad \text{(S)} \tag{8·2}$$

in the form of the Stratonovich-type, where

$$\alpha(x, t)=f(x, t)-\varepsilon g\frac{\partial g}{\partial x} \quad\text{and}\quad \beta(x, t)=g(x, t). \tag{8·3}$$

Consequently there is a possibility that a system can be reduced to an Itô-type equation with a linear drift term and consequently that the moment $\langle x(t)\rangle$ can be obtained explicitly. That is, it happens when $f(x, t)=\alpha+\varepsilon gdg/dx$ is linear in x.

In particular, the following model:

$$\frac{d}{dt}x(t)=ax+b+(cx^2+dx+e)^{1/2}\eta(t) \qquad \text{(S)} \tag{8·4}$$

can be transformed into an Itô-type equation with a linear drift term and with the use of the Itô formula all the moments $\langle x^n(t)\rangle$ are calculated explicitly, by noting that $\langle x^n(t)\eta(t)\rangle=0$. For example, we consider here the following simple case:

$$\dot{x}=a-bx+\sqrt{x}\,\eta(t) \qquad (\mathrm{S}). \tag{8·5}$$

The corresponding Itô-type equation is given by

$$\dot{x}=\left(a+\frac{1}{2}\varepsilon\right)-bx+\sqrt{x}\,\eta(t) \qquad (\text{Itô}). \tag{8·6}$$

The average $\langle x(t)\rangle$ is obtained as

$$\langle x(t)\rangle=\langle x(0)\rangle e^{-bt}+\frac{a+\frac{1}{2}\varepsilon}{b}(1-e^{-bt}), \tag{8·7}$$

by using the martingale property $\langle g(x)\eta(t)\rangle=0$. The slowing down occurs only at $b=0$, where $\langle x\rangle_{\mathrm{st}}=\infty$. This corresponds to the case (i) in the criterion. The stationary distribution function $P_{\mathrm{st}}(x)$ is given by

$$P_{\mathrm{st}}(x)=N_0x^{(a/\varepsilon)-(1/2)}\exp\left(-\frac{b}{\varepsilon}x\right). \tag{8·8}$$

Consequently, a phase transition occurs at $a=(1/2)\varepsilon$ as in Fig. 1. However, there occurs no critical slowing down in $\langle x(t)\rangle$ at $a=(1/2)\varepsilon$ in this system.

§ 9. Discussion

In this paper we have clarified the general criterion for the appearance of the slowing down and we have solved some multiplicative stochastic models with the use of the scaling theory as well as the method of formal solution. The strong-coupling expansion method is shown to be very powerful in some models which have a special kind of linear scaling property, to obtain exact relaxation spectra.

It is remarked that one should be careful about normalization conditions in solving stochastic processes or Fokker-Planck equations, as is discussed in Appendix B. We also note the appearance of slowing down depends on the initial distribution and the variable that we observe.

The mistake of widely-used normalization conditions is most clarified in the following model:

$$\frac{d}{dt}x(t)=-\gamma x+g+x\eta(t). \tag{9·1}$$

This equation is solved easily and we obtain

$$\langle x(t)\rangle=\frac{g}{\gamma-\varepsilon}+\left(x(0)-\frac{g}{\gamma-\varepsilon}\right)e^{-(\gamma-\varepsilon)t}. \tag{9·2}$$

Here, we note that $\langle x\rangle_{st}$ exists for $\gamma>\varepsilon$. We can solve the Fokker-Planck equation corresponding to (9·1) just in the same way in Appendix B. As we see in Appendix B, the spectrum $\lambda_1=(\gamma-\varepsilon)$ exists only for $\gamma>2\varepsilon$, when we impose Schrödinger-type normalization condition (B·1). Of course, the correct spectrum is obtained when we impose the correct normalization condition (B·2).

Extensions of the present treatment on a single macrovariable to systems of infinite degrees of freedom such as the TDGL model will be reported in a separate paper.

Acknowledgements

The present authors would like to thank Professor R. Kubo, Professor G. Nicolis, Dr. S. Kabashima, Dr. K. Kitahara, Dr. W. Horsthemke, Dr. R. Lefever and Mr. Y. Inaba for their stimulating discussions. This study is partially financed by the Mitsubishi Foundation.

Appendix A

——*Eigenvalue Problem of the SKS-Model*——

Fokker-Planck equation (1·3) is transformed by the variable $z\equiv\int\beta(x)^{-1}dx$ into

$$\frac{\partial\tilde{P}(z,t)}{\partial t}=-\frac{\partial}{\partial z}\gamma(z)\tilde{P}(z,t)+\varepsilon\frac{\partial^2}{\partial z^2}\tilde{P}(z,t), \tag{A·1}$$

where $\gamma(z)$ is defined as $\gamma(z)=\alpha(x(z))/\beta(x(z))$. This is the Fokker-Planck equation for additive noise and is transformed into the Schrödinger-type equation for the 'wave-function' $\phi=P_{st}{}^{-1/2}\tilde{P}$

$$-\varepsilon\frac{\partial\phi(z,t)}{\partial t}=\left(-\varepsilon^2\frac{\partial^2}{\partial z^2}+\frac{1}{4}(\gamma(z)^2+2\varepsilon\gamma'(z))\right)\phi(z,t). \tag{A·2}$$

Applying this method to the SKS-model, we obtain

$$-\varepsilon\frac{\partial\phi(z,t)}{\partial t}=\left\{-\varepsilon^2\frac{\partial^2}{\partial z^2}+\frac{1}{4}\left(\gamma^2(m-1)^2\left(z-\frac{g}{\gamma}\right)^2-2\varepsilon\gamma(m-1)\right)\right\}\phi \tag{A·3}$$

with the variable $z\equiv x^{1-m}$. This is the Schrödinger equation for the harmonic oscillator and the spectra are $\lambda_k=k\gamma(m-1)$.

Appendix B

——Normalization Condition of Stochastic Processes——

When we solve the Schrödinger-type equation (A·3), we have to note that the condition for the normalization is *not* that

$$|\phi(z)|^2 \quad \text{is integrable}\,, \tag{B·1}$$

but that

$$\tilde{P}(z)=\phi(z)P_{\rm st}^{1/2}(z)\equiv\phi(z)\phi_0(z) \quad \text{is integrable}\,. \tag{B·2}$$

This difference comes from the fact that the probability of a state is represented by $|\phi(z)|^2$ in quantum mechanics and that it is $\tilde{P}(z)$ in Fokker-Planck equations. When we take into account the boundary, the condition that the flux vanishes at the boundary is necessary. Then we have

$$J(z)=-\gamma(z)P(z)+\varepsilon\frac{\partial}{\partial z}P(z)=0\,. \tag{B·3}$$

We consider the Schenzle-Brand model to discuss this point in detail. The Schrödinger-type equation for $\psi(z)=\psi(\ln(g^{1/(m-1)}x))$ is obtained in the form

$$-\varepsilon\frac{\partial}{\partial t}\psi=\left[-\varepsilon^2\frac{\partial^2}{\partial z^2}+\left\{\frac{1}{4}(\gamma-e^{(m-1)z})^2-\frac{m-1}{2}\varepsilon e^{(m-1)z}\right\}\right]\psi\,, \tag{B·4}$$

in the same way as in Appendix A. This can be reduced to the Schrödinger equation in Morse potential and gives the following eigenfunctions and eigenvalues:

$$\psi_n=e^{-\xi/2}\xi^{S_n}F(-n,2S_n+1,\xi);\quad \xi=\frac{1}{\gamma\varepsilon}e^{(m-1)z}\,,$$

$$S_n=\frac{1}{m-1}\left(\frac{\gamma}{2\varepsilon}-n(m-1)\right),\quad \lambda_n=n(m-1)(\gamma-n(m-1)\varepsilon) \tag{B·5}$$

and continuous spectra for $\lambda>\gamma^2/(4\varepsilon)$. When we impose the condition (B·1), S_n must be positive, and Schenzle and Brand's result is reproduced.[5] However, in the correct condition (B·2) (the condition (B·3) gives the same condition in this model) we obtain $S_n+S_0>0$ and this condition results in the following condition that $n=0,1,\cdots,n_0''$ where n_0'' is the maximum integer value less than $\gamma/[\varepsilon(m-1)]$.

When there exists a mode that does not satisfy (B·1), (which we will call a 'divergent mode'), the usual method of eigenfunction expansion is not applicable. It is formulated as follows:

$$P(x,t)=\sum a_n\psi_n(x)\psi_0(x)e^{-\lambda_n t};\quad \psi_0(x)=P_{\rm st}^{1/2}(x), \tag{B·6}$$

and

$$a_n=\int\psi_n(x)\psi_0(x)P_{\rm ini}(x)dx\ ,$$

as is well-known. This comes from the orthonormality

$$\int\psi_m(x)\psi_n(x)dx=\delta_{mn}\ . \tag{B·7}$$

However, since $\int\psi_n{}^2(x)dx$ diverges for divergent modes ψ_n, we cannot use this method. Of course, we can determine a_n by defining such inner products that do not diverge for all modes, but it is rather difficult in practice, because orthogonality is not satisfied for such inner products.

Here, we remark that appearance of divergent modes depends on initial conditions and variables that we observe. If the initial distribution $P_{\rm ini}(x)$ does not include divergent modes, then $P(x,t)$ does not include them. Therefore, if

$$\int P_{\rm ini}^2(x)P_{\rm st}^{-1}(x)dx<\infty \tag{B·8}$$

holds, we do not need divergent modes.

However, when we study variables $F(x)$ that violate the condition

$$\int\psi_n(x)\psi_0(x)F(x)dx<\infty \tag{B·9}$$

for divergent modes $\{\psi_n\}$, we need divergent modes to express $\langle F(x)\rangle_t$, even if (B·8) is satisfied. This is derived as follows. Since

$$\langle F(x)\rangle_t=\sum_n\left(\int\psi_n(x)\psi_0(x)F(x)dx\right)a_ne^{-\lambda_n t}\ , \tag{B·10}$$

the contribution of divergent modes $\{\psi_n\}$ remains finite even if $a_n=0$, because the integration in (B·10) diverges for such modes.

For example, we consider the Schenzle-Brand model and put $F(x)=x^{1-m}$. Then it is easily shown that $\int F(x)\psi_0(x)\psi_1(x)dx$ diverges for $\gamma<2(m-1)\varepsilon$. When we exclude divergent modes, the spectrum $\lambda_1=(m-1)(\gamma-(m-1)\varepsilon)$ appears only if $\gamma>2(m-1)\varepsilon$[5] due to $S_1>0$. However, to describe the relaxation of $F(x)$, we always need the mode λ_1 if $\gamma>(m-1)\varepsilon$ (this condition is imposed from the requirement $\langle F(x)\rangle<\infty$), as we have already seen in §§ 2 and 3.

Appendix C

——*Derivation of the Spectra of the Model*
$\dot{x}=\gamma x^l-gx^{2m-1}+x^m\eta(t)$ for $l=1$ and $l=m$——

a) The case that $\dot{x}=\gamma x-gx^{2n-1}+x^n\eta(t)$. This equation is transformed into the Langevin equation

$$\dot{z}=\frac{g(n-1)}{z}-\gamma(n-1)z+\eta'(t);$$
$$\langle\eta'(t)\eta'(t')\rangle=2\varepsilon(n-1)^2\delta(t-t') \tag{C·1}$$

with $z=\int\beta^{-1}(x)dx$ and this is equivalent to the Schrödinger-type equation

$$-\varepsilon'\frac{\partial}{\partial t}\psi=\left[-\varepsilon'^2\frac{\partial^2}{\partial z^2}+V(z)\right]\psi(z,t), \tag{C·2}$$

where $\varepsilon'=(n-1)^2\varepsilon$ and

$$V(z)=\frac{1}{4}(n-1)g((n-1)g-2\varepsilon')\frac{1}{z^2}$$
$$+\frac{\gamma^2}{4}(n-1)^2z^2-\frac{1}{2}(n-1)\gamma((n-1)g+\varepsilon'). \tag{C·3}$$

This is the Schrödinger equation for the three-dimensional harmonic oscillator and the spectra of this system are given by $\lambda_k=2k\gamma(n-1)$.

These spectra are also obtained by the scaling theory. From (4·7), we get

$$\xi_{sc}(t)^{-(n-1)}=-(n-1)\int_0^t e^{(n-1)\gamma s}\eta(s)ds+x(0)^{-(n-1)}. \tag{C·4}$$

Using (4·10), we have

$$\langle x(t)\rangle_{sc}=\frac{g}{\gamma}b(t)\sum_k\binom{-\frac{1}{2(n-1)}}{k}\left(\frac{g}{\gamma}b(t)\right)^{-k}\langle\xi_{sc}(t)^{-2k(n-1)}\rangle e^{-2k(n-1)\gamma t} \tag{C·5}$$

with $b(t)=1-\exp(-2(n-1)\gamma t)$. Since $\langle\xi_{sc}(t)^{-2k(n-1)}\rangle$ is expressed by a polynomial of the quantity

$$\langle\xi_{sc}(t)^{-2(n-1)}\rangle=x(0)^{-2(n-1)}+\frac{\varepsilon}{\gamma}(n-1)(e^{2(n-1)\gamma t}-1), \tag{C·6}$$

we get the exact spectra λ_k given by (C·5).

b) The case that $\dot{x}=\gamma x^n-gx^{2n-1}+x^n\eta(t)$. Using the same method as in the case a), we obtain the following Schrödinger-type equation:

$$-\varepsilon'\frac{\partial}{\partial t}\psi=\left\{-\varepsilon'^2\frac{\partial^2}{\partial z^2}+\frac{1}{2}(n-1)^2\gamma^2\right.$$
$$\left.-\frac{1}{4}\frac{(n-1)^2}{z}g\gamma+\frac{(n-1)g}{4z^2}((n-1)g-2\varepsilon')\right\}\psi. \tag{C·7}$$

This is the Schrödinger equation for the three-dimensional Coulomb potential and we obtain the spectra

$$\lambda_k=\frac{\gamma^2}{4\varepsilon}\{1-[1+2\varepsilon k(n-1)/g)]^{-2}\}. \tag{C·8}$$

Of course, we have to be careful about the difference between the normalization of the Fokker-Planck and Schrödinger equations, but the spectra do not change in the cases treated in this appendix.

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