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Critical Slowing Down in Stochastic Processes. I

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The present series of papers clarify the physical mechanism of a critical slowing down in stochastic processes. It is classified into three categories, namely (i) deterministic, (ii) marginal and (iii) noise-induced critical slowing downs. In this paper, some general aspects of critical slowing down are studied in multiplicative stochastic processes. The value of the long-time exponent is derived on the basis of the dynamical scaling theory for general nonlinear stochastic processes as well as conditions for the appearance of the noise-induced long-time tail. Some exactly soluble examples are discussed.

§ 1. Introduction

One of the most fundamental features in critical dynamics is the phenomenon of critical slowing down, which means that the relaxation of the relevant order parameter becomes very slow near the critical point. The simplest phenomenological explanation of this phenomenon is given by the Landau-van Hove theory as follows. The temporal evolution of the order parameter, say $m(t)$, may be described by

$$\frac{d}{dt}m(t)=-\Gamma\frac{\partial F}{\partial m}\,;\quad F=F_0+\frac{1}{2}\gamma_0 m^2+\frac{1}{4}g_0 m^4+\cdots \tag{1·1}$$

in the mean field approximation. That is, we have

$$\frac{d}{dt}m(t)=\gamma m(t)-gm(t)^3 \tag{1·2}$$

with $\gamma=-\Gamma\gamma_0$ and $g=\Gamma g_0$. Here, Γ denotes a bare kinetic coefficient, and $\gamma_0 \propto T-T_c$. The solution of (1·2) is given by[1)]

$$m(t)=m(0)e^{\gamma t}\{1+\gamma^{-1}gm(0)^2(e^{2\gamma t}-1)\}^{-1/2}\,. \tag{1·3}$$

If $T>T_c$, i.e., $\gamma<0$, then the asymptotic behavior of $m(t)$ is given by $m(t) \sim\exp(-t/\tau)$, where $\tau=|\gamma|^{-1}\propto(T-T_c)^{-1}$, as was discussed in a previous paper.[2)] At the critical point $\gamma=0$, $m(t)$ shows the following long-time tail:

$$m(t)=m(0)\{1+2gm(0)^2t\}^{-1/2}\sim t^{-1/2}\,. \tag{1·4}$$

Practically, this long-time tail is observed in the region $\gamma t\ll 1$ and the exponential decay is observed for $\gamma t\gg 1$. Thus, the crossover effect occurs in the time region $t\sim\gamma^{-1}$. Namely, the crossover time becomes larger and larger, as the temperature approaches the critical point T_c. The above arguments are all based on the mean field theory,[1] in which fluctuations are neglected. In fact, due to the fluctuation effect the exponent Δ of critical slowing down is, in general, larger than[3] the susceptibility exponent which is usually bigger than unity.

In the present paper, we discuss the fluctuation effect due to a random force on the critical slowing down or long-time tail, by the use of nonlinear Langevin equations of the form

$$\frac{d}{dt}x = f(x) + g(x)\eta(t). \tag{1·5}$$

Here, $\eta(t)$ is a random force which is, for simplicity, assumed to be Gaussian and white, that is,

$$\langle \eta(t)\eta(t')\rangle = 2\varepsilon\delta(t-t'). \tag{1·6}$$

Throughout this series of papers, the stochastic differential equation is interpreted as the symmetrized one,[4] whose calculus is the same[4] as the ordinary one.

In the previous paper,[2] we defined the phase transition point γ_p at which the profile of the stationary distribution function changes drastically,*) and we discussed the slowing down of relaxation at the phase transition point. There is, however, another interesting point of the relevant parameter space γ contained in $f(x)$, that is, a critical point at which the system begins to change from a stable state to an unstable one in the deterministic limit $\varepsilon = 0$. We call this a *deterministic critical point.* What happens at this deterministic critical point in the presence of a random force? Does the critical slowing down appear at the deterministic critical point? Dekker and van Kampen[5] studied numerically the minimum (in magnitude) non-zero eigenvalue of the Fokker-Planck operator corresponding to the following nonlinear Langevin equation:

$$\frac{d}{dt}x = \gamma x - gx^3 + \eta(t). \tag{1·7}$$

They found that the minimum eigenvalue λ_1 is non-vanishing even at the deterministic critical point $\gamma = 0$, and consequently that the relaxation time $\tau \propto \lambda_1^{-1}$ is finite at $\gamma = 0$, i.e., there occurs no critical slowing down for $\varepsilon \neq 0$. On the other hand, Brenig and Banai[6] studied the following multiplicative stochastic process:[2),7)~10)]

$$\frac{d}{dt}x = \gamma x - gx^3 + x\eta(t), \tag{1·8}$$

using Carleman's method,[11)~14)] and they found the following noise-induced long-time tail:

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*) Figure 1 in Ref. 2) has a simple error. $\gamma > \gamma_p$ in (b) should read $\gamma < \gamma_p$.

$$\langle x^2(t)\rangle \sim (\varepsilon/g^2 t)^{1/2} \sim t^{-1/2}, \tag{1·9}$$

at $\gamma=0$. This is clearly more dominant than the deterministic critical slowing down $\langle x^2(t)\rangle_{\varepsilon=0}=x^2(0)\{1+2gx^2(0)t\}^{-1}\sim t^{-1}$ which is immediately obtained from (1·4). This is an intrinsic noise-induced long-time tail. Now arise some interesting questions why it happens and what is the general condition for the appearance of long-time tail. To answer these questions is the purpose of this series of papers.

Without loss of generality, the stationary point x_e in (1·5) can be put zero at the critical point, because a nonvanishing x_e is shifted by a simple transformation $x-x_e=x'$ and because $x_e=\infty$ can be mapped into $x_e'=0$ by the transformation $x'=1/x$. Thus, hereafter we assume that $x_e=0$ is the final stationary point of the system (1·5) at the critical point and that all the moments $\langle x^p(t)\rangle$ for $p>0$ go to zero in power laws for $t\to\infty$ at the critical point. The condition of this second assumption will be discussed in detail in the succeeding sections. As we are interested in the asymptotic behavior of the solution of (1·5), we may retain only the dominant terms as follows:

$$f(x)=\gamma x^l - gx^m + o(x^m) \quad \text{and} \quad g(x)=x^n+o(x^n), \tag{1·10}$$

where $g>0$, $n>0$ and $m>l>0$. Since we assume that $\langle x^p(t)\rangle\to 0$ as $\gamma\to 0$, the noise has no additive part (i.e., $g(0)=0$). In the general situation where $g(0)\neq 0$, our arguments hold in the time region where $\langle x^n(t)\rangle \gg g(0)$, after which the crossover to the system with the additive noise occurs. More mathematically the above asymptotic forms (1·10) can be obtained in a generalized singular perturbation method.[15] Thus, our skeletonized nonlinear Langevin equation is[2]

$$\frac{d}{dt}x=\gamma x^l - gx^m + x^n\eta(t). \tag{1·11}$$

The case $l=n=1$ has been discussed already by several authors[7),9),10)] in connection with chemical reaction for fluctuating reaction rate and with laser amplification for fluctuating pumping rate. To discuss the physical meaning of the nonlinear Langevin equation (1·11), we first consider the simple example (1·7). If there exists no random force, i.e., $\varepsilon=0$, then Eq. (1·7) expresses the temporal evolution equation of the macroscopic variable x which is obtained in the mean field approximation.[1] Interaction of x with the remaining infinite degrees of freedom may be taken into account approximately by introducing a random force $\eta(t)$. Secondly when the growing-rate γ is fluctuating in (1·2), we obtain the multiplicative stochastic process (1·8).

As Mori[16] has shown, the true Langevin equation projected from the relevant original Liouville equation has a memory effect and its random force is, in general, very complicated. Our nonlinear Langevin equation (1·5) or (1·11) may be regarded as the simplest idealization for a single macrovariable.

In § 2, the scaling property of the nonlinear Langevin equation is studied and the relation between the critical slowing down and the singularity of moments in the stationary state is derived on the basis of the dynamical scaling law.[17] In § 3, we clarify three different kinds of mechanism of critical slowing down in multiplicative stochastic processes, and we obtain explicitly the value of the critical exponent of slowing down in each situation. In particular, the physical mechanism of the noise-induced long-time tail is explained in general situations under some appropriate conditions. In § 4, conditions for the appearance of long-time tail are studied in detail, on the basis of the general criterion of appearance of the slowing down and also by the help of the Schrödinger-type equation corresponding to (1·5). Summary and discussion are given in § 5.

§ 2. Dynamical scaling law and critical slowing down

We study first the scaling property of the nonlinear stochastic process (1·11) by introducing the following linear scaling transformation:

$$g'=\zeta^{\alpha}g\,,\quad t'=\zeta^{\beta}t\,,\quad \varepsilon'=\zeta^{-1}\varepsilon\,,\quad x'=\zeta^{\delta}x\,,\quad \gamma'=\zeta^{\nu}\gamma \tag{2·1}$$

with[2),*)]

$$\beta=\frac{m-1+2\alpha(n-1)}{m-2n+1}\,,\quad \delta=\frac{-(\alpha+1)}{m-2n+1}\,,\quad \nu=\frac{(l-2n+1)\alpha-(m-l)}{m-2n+1}\,. \tag{2·2}$$

Using that the Gaussian white noise $\eta(t)$ satisfying (1·6) has the same scaling property as $(\varepsilon/t)^{1/2}$, we can easily show that (1·11) is invariant for the above linear scaling transformation (2·1) with (2·2).

Hereafter we treat the case $l=1$, which corresponds to a linear growing-rate model. For this case, we find that

$$\gamma't'=\gamma t \quad\text{and}\quad \frac{\gamma'}{\varepsilon'}\left(\frac{g'}{\gamma'}\right)^{2(n-1)/(m-1)}=\frac{\gamma}{\varepsilon}\left(\frac{g}{\gamma}\right)^{2(n-1)/(m-1)}, \tag{2·3}$$

namely that these two quantities are independent dimensionless invariants for the transformation (2·1). It is easily shown that there is no other independent invariant. Therefore, such a part $\langle Q(t)\rangle$ of any moment $\langle x^{p}(t)\rangle$ of the solution $x(t)$ in (1·11) that does not depend on the initial value has the following form:

$$\langle Q(t)\rangle/\langle Q\rangle_{\rm st}=f_{Q}\left(\gamma t,\frac{\gamma}{\varepsilon}\left(\frac{g}{\gamma}\right)^{2(n-1)/(m-1)}\right), \tag{2·4}$$

where $\langle Q\rangle_{\rm st}$ denotes the stationary value of Q. Therefore if $\langle Q(t)\rangle$ does not depend on the nonlinearity g, then it is a function only of γt.

*) The second equation of Eq. (5·3) in Ref. 2) should read $y=-(\alpha+1)/(m-2n+1)$.

Next, we study the asymptotic behavior of $\langle Q(t)\rangle$ near the critical point $\gamma=0$. We assume here the following dynamical scaling law:

$$\langle Q(t)\rangle/\langle Q\rangle_{st}\simeq f_Q^{(sc)}(\gamma^{\Delta}t/\tau_0), \tag{2·5}$$

where τ_0 is a dimensional parameter to make the scaling variable $\gamma^{\Delta}t/\tau_0$ dimensionless, and it may depend on g and ε. Here Δ denotes the dynamical critical exponent to express the time scale near the critical point $\gamma=0$. If $\langle Q\rangle_{st}$ takes the following asymptotic form

$$\langle Q\rangle_{st}\sim\gamma^{\varphi_Q}, \tag{2·6}$$

then we have

$$\begin{aligned}\langle Q(t)\rangle &\sim \gamma^{\varphi_Q}f_Q^{(sc)}(\gamma^{\Delta}t/\tau_0)\\ &\sim t^{-\varphi_Q/\Delta}g_Q^{(sc)}(\gamma^{\Delta}t/\tau_0)\sim t^{-\psi_Q}\end{aligned} \tag{2·7}$$

with $g_Q^{(sc)}(x)=x^{\varphi_Q/\Delta}f_Q^{(sc)}(x)$. Thus, we arrive at the following scaling relation:

$$\psi_Q=\varphi_Q/\Delta\ . \tag{2·8}$$

The above relations (2·6)~(2·8) are also applicable to the case that $\psi_Q<0$ and $\varphi_Q<0$, in which $\langle Q\rangle_{st}$ and $\langle Q(t)\rangle$ diverge as $\gamma\to0$ and $t\to\infty$, respectively. We may assume that the present stochastic system (1·11) has a unique time scale independent of physical quantities such as $\{x^p(t)\}$ near the critical point, namely that Δ is universal irrespectively of Q or p.

More explicitly we consider a moment of the form $\langle x^p(t)\rangle$ and we study the corresponding two exponents φ_p and ψ_p. The scaling relation (2·8), i.e., $\psi_p=\varphi_p/\Delta$ yields the statement that the p-dependence of the long-time tail exponent ψ_p is the same as that of the static exponent φ_p. This proposition will be very useful in the succeeding section. In particular, this relation will be used to obtain the critical value p_c above which ψ_p does not depend on p.

It will also be interesting to note some invariant properties for the following nonlinear transformation:

$$y=x^q\ . \tag{2·9}$$

Applying (2·9) to (1·11), we obtain the following transformed equation:

$$\frac{dy}{dt}=\gamma' y-g' y^{m'}+y^{n'}\eta'(t) \tag{2·10}$$

with $\gamma'=q\gamma$, $g'=qg$, $\eta'(t)=q\eta(t)$ and

$$m'=(m-1+q)/q\ ,\qquad n'=(n-1+q)/q\ . \tag{2·11}$$

The quantity Q_1' defined by $Q_1'\equiv q\{m'-(2n'-1)\}$ is invariant for the above nonlinear transformation (2·9), that is,

$$Q_1' \equiv q\{m'-(2n'-1)\} = m-(2n-1) \equiv Q_1 . \tag{2·12}$$

There is another interesting invariant quantity concerning the moment $\langle x^p(t)\rangle = \langle y^{p'}(t)\rangle$ with $p'=p/q$, namely

$$Q_2' \equiv q\{p'-(n'-1)\} = p-(n-1) \equiv Q_2 . \tag{2·13}$$

Consequently the sign of $p-(n-1)$ is invariant for the above nonlinear transformation. This property will also be useful in § 3.

The nonlinear Langevin equation (1·11) for $l=1$ with (1·6) is equivalent to the following hierarchy of equations of moments

$$\frac{d}{dt}\langle x^p(t)\rangle = p\gamma\langle x^p(t)\rangle - pg\langle x^{m+p-1}(t)\rangle + p(n+p-1)\varepsilon\langle x^{2n+p-2}(t)\rangle , \tag{2·14}$$

where we have made use of Novikov's theorem.[18),4)] The above equations of motion (2·14) will be discussed in detail in § 3.

§ 3. Critical slowing down in multiplicative stochastic processes

Now under the assumption that all the moments $\langle x^p(t)\rangle$ for some positive range of p go to zero in power laws for $t\to\infty$ at the critical point, we discuss the long-time tail of $\langle x^p(t)\rangle$ in the nonlinear Langevin equation

$$\frac{d}{dt}x = \gamma x - gx^m + x^n\eta(t) \tag{3·1}$$

at the critical point $\gamma=0$, namely in the "critical equation"

$$\frac{d}{dt}x = -gx^m + x^n\eta(t). \tag{3·2}$$

We are interested in the long-time tail exponent ψ_p defined by

$$\langle x^p(t)\rangle \simeq t^{-\psi_p} \tag{3·3}$$

at $\gamma=0$ for $t\to\infty$. There are three different situations, namely, (i) deterministic, (ii) marginal and (iii) noise-induced long-time tails. Each long-time tail appears in the following situations:

$$\text{(i)}\quad A=B\gg C , \tag{3·4}$$

$$\text{(ii)}\quad A=B=C , \tag{3·5}$$

$$\text{(iii)}\quad A\ll B=C \quad\text{or}\quad B\ll A=C , \tag{3·6}$$

where A, B and C denote the scaling dimensionality in time for $d\langle x^p(t)\rangle/dt$, $g\langle x^{m+p-1}(t)\rangle$ and $\langle x^{n+p-1}\eta(t)\rangle$, respectively, namely

$$A \simeq t^{-(\psi_p+1)}, \qquad B \simeq t^{-\psi_{m+p-1}}, \qquad C \simeq t^{-\psi_{2n+p-2}}. \tag{3·7}$$

(i) *Deterministic long-time tail* ($m<2n-1$)

If the noise term is asymptotically smaller than the deterministic term in (3·2), namely if (3·4) is satisfied, then we obtain the deterministic long-time tail

$$\langle x^p(t)\rangle \simeq t^{-\psi_p}; \qquad \psi_p = \frac{p}{m-1} \tag{3·8}$$

from the balancing condition that $A=B$, that is,

$$\psi_p + 1 = \psi_{m+p-1}. \tag{3·9}$$

The condition that $B \gg C$ gives the inequality

$$m < 2n-1 \tag{3·10}$$

together with $m>1$.

The stationary moment $\langle x^p\rangle_{st}$ takes the following asymptotic form:

$$\langle x^p\rangle_{st} \simeq (\gamma/g)^{p/(m-1)} \tag{3·11}$$

near the critical point $\gamma=0$. Thus, the static exponent φ_p is given by

$$\varphi_p = \frac{p}{m-1}. \tag{3·12}$$

Therefore, the dynamic critical exponent Δ is found to be equal to unity (i.e., $\Delta=1$) through the scaling relation (2·8), namely $\psi_p = \varphi_p/\Delta$.

The simplest example of the deterministic case is the deterministic limit $\varepsilon=0$ such as (1·2). Another exactly soluble example is the SKS-model[2)]

$$\frac{dx}{dt} = \gamma x - gx^m + x^m \eta(t) \tag{3·13}$$

for $m>1$. It is easily shown that

$$\langle x^p(t)\rangle \simeq \{(m-1)gt\}^{-p/(m-1)} \tag{3·14}$$

for $p<m-1$. Clearly we have $\langle x^p\rangle_{st} \cong (\gamma/g)^{p/(m-1)}$. For more details, see Appendix A. The restriction that $p<m-1$ can be removed, if we include terms of higher order in (3·1). For details, see Appendix B.

(ii) *Marginal case* (*i.e.*, $m=2n-1$)

In the case $m=2n-1$, the nonlinear term in (2·14) becomes of the same order as the noise term in (2·14). The condition (3·5) yields

$$\psi_p+1=\psi_{m+p-1}=\psi_{2n+p-2}\,. \tag{3·15}$$

A monotonic solution of (3·15) is

$$\psi_p=\frac{p}{m-1}\,. \tag{3·16}$$

This case corresponds to the following model:

$$\frac{d}{dt}x=\gamma x-gx^{2n-1}+x^n\eta(t). \tag{3·17}$$

As was discussed in the previous paper,[2] this model can be solved rigorously by using the eigenfunction expansion method as

$$\langle x(t)\rangle=\sum_{m=0}^{\infty}\frac{\varphi_m(x_0)}{\varphi_0(x_0)}e^{-\lambda_m t}\int\varphi_m(x)x\varphi_0(x)dx \tag{3·18}$$

with an eigenfunction $\varphi_m(x)$ for the eigenvalue $\lambda_m=2m\gamma(n-1)$. An explicit expression of (3·18) is given by

$$\langle x(t)\rangle=\left(\frac{\gamma}{2\varepsilon N}\right)^{1/2N}\frac{\Gamma(a+1-1/2N)}{\Gamma(1/2N)}\sum_{m=0}^{\infty}\frac{\Gamma(m+1/2N)}{\Gamma(m+a+1)}L^a{}_m\left(\frac{\gamma x_0^{-2N}}{2\varepsilon N}\right)e^{-2m\gamma Nt}\,, \tag{3·19}$$

where $a=g/(2\varepsilon N)-\frac{1}{2}$, $N\equiv n-1$ and $L_n{}^a(x)$ is a Laguerre polynomial.

Equation (3·19) can be resummed in such a more convenient form as

$$\langle x(t)\rangle=\left(\frac{\gamma}{2N\varepsilon}\frac{1}{(1-e^{-2\gamma Nt})}\right)^{1/2N}\frac{\Gamma\left(a+1-\dfrac{1}{2N}\right)}{\Gamma(a+1)}\,{}_1F_1\left(\frac{1}{2N};a+1;\frac{y_0}{1-e^{2N\gamma t}}\right) \tag{3·20}$$

with $y_0=\gamma x_0^{-2N}/(2\varepsilon N)$. Here, we have used the formula

$${}_1F_1(x;y;z)=\sum_{k=0}^{\infty}\frac{\Gamma(k+x)}{\Gamma(k+y)}\frac{\Gamma(y)}{\Gamma(x)}\frac{z^k}{k!}\,. \tag{3·21}$$

The above expression (3·20) leads to the long-time tail of the form $\langle x(t)\rangle\sim t^{-1/[2(n-1)]}$ at the critical point $\gamma=0$.

The above result (3·20) is a special case of the following more general result:

$$\langle x^p(t)\rangle=\left(\frac{\gamma}{2N\varepsilon}\frac{1}{1-e^{-2N\gamma t}}\right)^{p/2N}\frac{\Gamma\left(a+1-\dfrac{p}{2N}\right)}{\Gamma(a+1)}$$
$$\times{}_1F_1\left(\frac{p}{2N};a+1;\frac{y_0}{1-e^{2N\gamma t}}\right), \tag{3·22}$$

which is calculated from a compact solution of the distribution function $P(x,t)$.

For the derivation of (3·22), see Appendix C. Clearly, we have

$$\langle x^p\rangle_{\rm st}=\left(\frac{\gamma}{2\varepsilon(n-1)}\right)^{p/2(n-1)}\frac{\Gamma\left(a+1-\frac{p}{2N}\right)}{\Gamma(a+1)}. \tag{3·23}$$

Thus, we get $\varphi_p=p/\{2(n-1)\}$ for $p<n-1+g/\varepsilon$. The long-time behavior of $\langle x^p(t)\rangle$ is given by

$$\langle x^p(t)\rangle\simeq t^{-\psi_p}; \qquad \psi_p=\frac{p}{2(n-1)} \tag{3·24}$$

at $\gamma=0$ for $p<n-1+g/\varepsilon$. This indicates that $\Delta=1$, i.e., $\psi_p=\varphi_p$.

This long-time tail appears even for $\varepsilon=0$ or $g\to 0$. This is the physical meaning of the "marginal case".

(iii) *Noise-induced long-time tail.* ($m>2n-1$)

When the noise in (3·1) is more dominant than the nonlinear term in (3·1) or the former is balancing with the latter, we call the long-time tail a noise-induced one. As $\psi_p\geq\psi_q$ for $p\geq q>0$, this noise-induced long-time tail appears when $m>2n-1$, because $\psi_{m+p-1}\geq\psi_{2n+p-2}$ in this case. It is more convenient to divide our arguments into the following two cases, namely $n=1$ and $n>1$.

(iii-a) $n=1$. This corresponds to the random growing-rate model

$$\frac{d}{dt}x=(\gamma+\eta(t))x-gx^m. \tag{3·25}$$

The stationary distribution function[2] $P_{\rm st}(x)$ is given by[7),2)]

$$P_{\rm st}(x)=Cx^{(\gamma-\varepsilon)/\varepsilon}\exp\left\{-\frac{g}{\varepsilon(m-1)}x^{m-1}\right\}. \tag{3·26}$$

Therefore, the stationary value of the moment takes the following asymptotic form:

$$\langle x^p\rangle_{\rm st}=\left(\frac{(m-1)\varepsilon}{g}\right)^{p/(m-1)}\Gamma\left(\frac{\gamma+p\varepsilon}{(m-1)\varepsilon}\right)\Big/\Gamma\left(\frac{\gamma}{(m-1)\varepsilon}\right)$$

$$\simeq\gamma \qquad \text{for } \gamma\to 0. \tag{3·27}$$

Thus, we have $\varphi_p=1$ for any positive value of p. The time scale exponent Δ is estimated as follows. In (3·25), γ and $\eta(t)$ play the same role with respect to the time scale. On the other hand we have

$$\langle\eta(\lambda t)\eta(\lambda t')\rangle=\frac{2\varepsilon}{\lambda}\delta(t-t')=\frac{1}{\lambda}\langle\eta(t)\eta(t')\rangle. \tag{3·28}$$

Consequently, we obtain

$$\eta(\lambda t)=\frac{1}{\sqrt{\lambda}}\eta(t). \tag{3·29}$$

Therefore, the quantity $\gamma t^{1/2}$ is the scaling variable in the model (3·25). That is, we obtain

$$\Delta=2\,. \tag{3·30}$$

The dynamical scaling relation (2·8) yields

$$\langle x^p(t)\rangle \simeq t^{-\psi_p}\,; \qquad \psi_p=\varphi_p/\Delta=\frac{1}{2} \tag{3·31}$$

for any positive value of p. This is a quite remarkable result in comparison with the fact that ψ_p is proportional to p in the other two cases (i) and (ii).

The result (3·31) can also be obtained rigorously from the following formal solution[8),2)] of (3·25):

$$x(t)=\frac{x_0\exp[\gamma t+W(t)]}{\left[1+gx_0{}^M M\int_0^t \exp\{M(\gamma t'+W(t'))\}dt'\right]^{1/M}}\,, \tag{3·32}$$

where $M\equiv m-1$, $x_0\equiv x(0)$ and

$$W(t)=\int_0^t \eta(t')dt'\,. \tag{3·33}$$

The detailed derivation of (3·31) from (3·32) will be reported in the second paper of the present series.[19)]

It is of great interest to try to understand the intuitive and physical mechanism of the noise-induced long-time tail, particularly the unexpected result that $\langle x^p(t)\rangle\sim t^{-1/2}$ for $p>0$ in the random growing-rate model. This phenomenon occurs due to the balancing of the noise term and nonlinear term in (3·1). More explicitly we consider the balancing condition

$$\psi_{m+p-1}=\psi_{2n+p-2} \tag{3·34}$$

in the equation of moments (2·14). A monotonic solution of (3·34) is given by ψ_k $=$constant$\equiv c(m)$ for $k>0$. This constant may depend on m. In order to evaluate this constant $c(m)$, we apply the nonlinear transformation (2·9) to get the relation

$$c(m)=c(m')=c(1+(m-1)/q) \tag{3·35}$$

with use of the relation (2·11) between m and m'. Note that $n'=n=1$. By taking the limit $q\to 0$ in (3·35), we arrive at

$$c(m)=c(\infty)=\text{independent of } m\,. \tag{3·36}$$

Therefore, we study now the limit $m\to\infty$. This corresponds to the following linear system:

$$\frac{dx}{dt}=\gamma x+x\eta(t); \qquad 0\leq x\leq 1 \tag{3·37}$$

with a reflecting wall at $x=1$. The solution of (3·37) with $\gamma=0$ is expressed by the following distribution function:

$$P(y,\,t)=\frac{1}{\sqrt{4\pi\varepsilon t}}\left[\exp\left\{-\frac{(y-y_0)^2}{4\varepsilon t}\right\}+\exp\left\{-\frac{(y+y_0)^2}{4\varepsilon t}\right\}\right] \tag{3·38}$$

with $y=\log x$ and $y_0=\log x_0$. The moment $\langle x^p(t)\rangle$ is given by

$$\begin{aligned}\langle x^p(t)\rangle&=\frac{1}{\sqrt{\pi}}e^{p^2\varepsilon t}\sum_{\sigma=\pm 1}e^{\sigma p y_0}\,\mathrm{Erfc}\left(\frac{\sigma y_0+2p\varepsilon t}{(4\varepsilon t)^{1/2}}\right)\\&=\frac{1}{\sqrt{\pi}}e^{-y_0^2/4\varepsilon t}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^{n+1}}\sum_{\sigma=\pm 1}\left\{\frac{(4\varepsilon t)^{1/2}}{2p\varepsilon t+\sigma y_0}\right\}^{2n+1}\\&\simeq\frac{1}{p(\pi\varepsilon t)^{1/2}}+O(t^{-3/2})\end{aligned} \tag{3·39}$$

at $\gamma=0$. Thus, we obtain $c(\infty)=\frac{1}{2}$. Therefore, we arrive at the conclusion that

$$\psi_p=c(\infty)=\frac{1}{2} \tag{3·40}$$

for $p>0$. It is also instructive to remark that the moment $\langle x^p(t)\rangle$ shows the following scaling behavior:

$$\begin{aligned}\frac{\langle x^p(t)\rangle}{\langle x^p\rangle_{\mathrm{st}}}&\simeq 1+\left(\frac{\varepsilon}{\pi\gamma^2 t}\right)^{1/2}\exp\left(-\frac{\gamma^2 t}{4\varepsilon}\right)-\frac{1}{\sqrt{\pi}}\,\mathrm{Erfc}\left(\sqrt{\frac{\gamma^2 t}{4\varepsilon}}\right);\\\langle x^p\rangle_{\mathrm{st}}&\simeq\frac{\gamma}{p\varepsilon},\end{aligned} \tag{3·41}$$

near the critical point (i.e., $\gamma\simeq 0$). This is a nice example of (2·4) for $n=1$ and also of (2·5) with $\Delta=2$.

The special case that $p=2$ and $m=3$ has been studied first by Brenig and Banai.[13] They obtained $\psi_2=1/2$. Our present investigation clarifies the physical reason of their result.

(iii-b) $n>1$. The stationary distribution function for the case $n>1$ is given by

$$P_{\mathrm{st}}(x)=Cx^{-n}\exp\left\{-\frac{\gamma x^{-2(n-1)}}{2(n-1)\varepsilon}-\frac{gx^{m-2n+1}}{(m-2n+1)\varepsilon}\right\}. \tag{3·42}$$

The stationary moment $\langle x^p\rangle_{\mathrm{st}}$ takes the following asymptotic forms:

$$\langle x^p\rangle_{\mathrm{st}}\simeq\left(\frac{\gamma}{2\varepsilon(n-1)}\right)^{p/2(n-1)}\Gamma\left(\frac{1}{2}-\frac{p}{2(n-1)}\right)\Big/\Gamma\left(\frac{1}{2}\right), \tag{3·43}$$

namely $\varphi_p = p/\{2(n-1)\}$ for $p < n-1$

$$\langle x^p \rangle_{st} \simeq \left(\frac{\gamma}{2\varepsilon(n-1)}\right)^{1/2} \frac{2(n-1)}{m-2n+1} \left(\frac{\varepsilon(m-2n+1)}{g}\right)^{-(n-p-1)/(m-2n+1)} \frac{\Gamma\left(\frac{p-n+1}{m-2n+1}\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{3·44}$$

namely $\varphi_p = 1/2$ for $p > n-1$, and

$$\langle x^p \rangle_{st} \simeq \frac{2(n-1)}{\Gamma\left(\frac{1}{2}\right)} \left\{\frac{\gamma}{2(n-1)\varepsilon}\right\}^{1/2} \log\left\{\frac{2(n-1)\varepsilon}{\gamma}\right\} \tag{3·45}$$

for $p = n-1$.

By the help of the dynamical scaling relation (2·8), we obtain

$$\langle x^p(t) \rangle \simeq t^{-\psi_p}; \qquad \psi_p = \varphi_p / \Delta , \tag{3·46}$$

where $\varphi_p = p/\{2(n-1)\}$ for $p < n-1$ and $\varphi_p = 1/2$ for $p > n-1$. It is interesting that there are two kinds of p-dependence of ψ_p, namely ψ_p is constant for $p > n-1$, and $\psi_p \propto p$ for $p < n-1$.

The value of Δ cannot be calculated phenomenologically and it remains a problem in future to evaluate explicitly the value of Δ for given values of m and n in the case of the noise-induced long-time tail for $n > 1$. As was discussed in § 2, we have $\Delta = 1$, if the scaling form (2·4) for the moment $\langle x^p(t) \rangle$ does not depend on the nonlinearity g.

The above situation is well understood if we study the limit $m \to \infty$. This corresponds to the following model:

$$\frac{d}{dt} x = \gamma x + x^n \eta(t); \qquad 0 \leq x \leq 1 \tag{3·47}$$

with a reflecting wall at $x = 1$. The solution of (3·47) for $\gamma = 0$ for $n > 1$ is given by

$$P(x, t) = \frac{x^{-n}}{(4\pi\varepsilon t)^{1/2}} \left\{\exp\left\{-\frac{(x^{-(n-1)} - x_0^{-(n-1)})^2}{4(n-1)^2 \varepsilon t}\right\} + \exp\left\{-\frac{(x^{-(n-1)} + x_0^{-(n-1)} - 2)^2}{4(n-1)^2 \varepsilon t}\right\}\right\}. \tag{3·48}$$

The moment $\langle x^p(t) \rangle$ takes the following asymptotic forms:

$$\langle x^p(t) \rangle \simeq \left\{\Gamma\left(\frac{1}{2} - \frac{p}{2(n-1)}\right) \Big/ \Gamma\left(\frac{1}{2}\right)\right\} (4(n-1)^2 \varepsilon t)^{-p/\{2(n-1)\}} \tag{3·49}$$

for $p < n-1$, and

$$\langle x^p(t) \rangle \simeq \log(\varepsilon t) / \{(n-1)(4\pi\varepsilon t)^{1/2}\} \tag{3·50}$$

for $p=n-1$, and

$$\langle x^p(t)\rangle \simeq \{(p-n+1)(\pi\varepsilon t)^{1/2}\}^{-1} \tag{3·51}$$

for $p>n-1$ at the critical point $\gamma=0$. Since $\varphi_p=p/\{2(n-1)\}$ for $p<n-1$ and $\varphi_p=1/2$ for $p>n-1$, we obtain $\Delta=1$ in the limit $m\to\infty$. If Δ does not depend on the nonlinearity exponent m, then we may have $\Delta=1$ even for a finite value of m in the case $n>1$. It should be noted that the boundary between the two regions (3·49) and (3·51) is given by $p=p_c\equiv n-1$. The sign of $p-(n-1)$ is invariant for the nonlinear transformation (2·9), as was discussed in § 2.

The long-time tail ψ_p is also derived for $n>1$ by the balancing condition that $A\ll B=C$ in (3·6), namely (3·34) in the case $p>n-1$ to give $\psi_p=1/2$ and by the balancing condition that $B\ll A=C$ in (3·6) for $p<n-1$ to give $\psi_p=p/\{2(n-1)\}$.

§ 4. Conditions for the appearance of long-time tail

In the preceding section we have clarified the physical mechanism of the long-time tail at the deterministic critical point, assuming the existence of the long-time tail. Now we study conditions for this existence in two different methods; namely on the basis of our general criterion[2)] of appearance of slowing down and using the spectrum of the Schrödinger-type equation corresponding to (3·1).

(i) *Conditions based on the previous general criterion of slowing down*

It will be instructive to repeat our general criterion[2)] of slowing down: *For a slowing down to appear, at least one physical mode Q should exist such that $\langle Q\rangle_{st}=\pm\infty$ at some point γ_0 or the stationary distribution $P_{st}(x)$ or the initial distribution $P_{ini}(x)$ should become unnormalizable at some point γ_0.*

A necessary condition of the appearance of long-time tail is that the relevant system satisfies the above criterion of slowing down at $\gamma\to 0$. We assume that the initial distribution is normalizable in our problem. Furthermore we are now interested in the case that all the moments $\langle x^p(t)\rangle$ for $p>0$ go to zero for $t\to\infty$, namely $\langle x^p\rangle_{st}=0$ (finite) at the critical point. Thus, our criterion of slowing down reduces in our problem to the requirement that $P_{st}(x)$ becomes singular at $\gamma=0$, namely $P_{st}(x)\propto\delta(x)$ as $\gamma\to +0$ in our situation. In order to study the condition for $P_{st}(x)$ to become a delta function $\delta(x)$, we make use of the Fokker-Planck equation[2),8)]

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}\Big[f(x)+\varepsilon g(x)g'(x)\Big]P+\varepsilon\frac{\partial^2}{\partial x^2}(g(x))^2P\,, \tag{4·1}$$

which corresponds to (1·5). The stationary distribution is given by[2)]

$$P_{st}(x)=N_0\exp\Big[\frac{1}{\varepsilon}\varphi(x)\Big];\quad \varphi(x)=-\varepsilon\log g(x)+\int^x\frac{f(y)}{(g(y))^2}dy\,, \tag{4·2}$$

where N_0 is the normalization constant. From this expression, we find easily that $P_{st}(x)$ becomes a delta function, as $\gamma\to+0$ in the cases (i) $2n>m+1>2$, (ii) $m+1>2n\geq 2$ and (iii) $m+1=2n\geq 2$ with additional conditions $l<2n-1$ or $l=n=1$.

(ii) *Conditions based on the distribution of spectra in the Schrödinger-type equation*

As was discussed in previous papers,[2),7),8),10)] the Fokker-Planck equation (4·1) can be expressed as the following Schrödinger-type equation:

$$-\frac{\partial\varphi(z,t)}{\partial t}=\left\{-\varepsilon\frac{\partial^2}{\partial z^2}+V(z)\right\}\varphi(z,t), \tag{4·3}$$

where*

$$V(z)=\frac{1}{4\varepsilon}\{\gamma(z)^2+2\varepsilon\gamma'(z)\},\quad \gamma(z)=f(x(z))/g(x(z)) \tag{4·4}$$

and the new variable z is introduced through $z\equiv\int^x g(y)^{-1}dy$. Several explicit expressions of $V(z)$ have been given for (1·11) in the previous paper.[2)] If it is possible to calculate all the eigenvalues and eigenfunctions of the above Schrödinger-type operator, then we can get any information about the temporal evolution of the moments $\langle x^p(t)\rangle$. In fact, the temporal evolution of a physical quantity Q is expressed in the form

$$\langle Q(t)\rangle-\langle Q\rangle_{st}=\int_{\lambda_0}^{\infty}e^{-\lambda t}\rho_Q(\lambda)d\lambda \tag{4·5}$$

with $\lambda_0\geq 0$, where $\rho_Q(\lambda)$ is the spectral density corresponding to Q. When $\lambda_0>0$ even for $\gamma=0$, no long-time tail appears. If $\lambda_0=0$ for $\gamma=0$ and $\rho_Q(\lambda)$ has the spectral edge like $\rho_Q(\lambda)\sim\lambda^a$ near $\lambda=0$ for $\gamma=0$, then we have $\langle Q(t)\rangle\sim t^{-(a+1)}$, if $\langle Q\rangle_{st}\to 0$. However, it is, in general, difficult to calculate the spectral density $\rho_Q(\lambda)$ explicitly. We are here satisfied with the discussion about the condition of long-time tail. From the above general argument, if the spectrum is continuous up to $\lambda=0$ (i.e., $\lambda_0=0$) for $\gamma=0$, it is expected that the long-time tail appears for any physical quantity Q. In this reason, we study the asymptotic behavior of the effective potential $V(z)$:

$$V(z)=\frac{1}{4\varepsilon}(\gamma e^{Lz}-ge^{Mz})^2+\frac{1}{2}(\gamma Le^{Lz}-gMe^{Mz}) \tag{4·6}$$

for $n=1$ (see Fig. 1), and

$$V(z)=\frac{N^2}{4\varepsilon}z^2(\gamma z^{-L/N}-gz^{-M/N})^2 -\frac{N^3}{2}\left\{\gamma\left(1-\frac{L}{N}\right)z^{-L/N}-g\left(1-\frac{M}{N}\right)z^{-M/N}\right\} \tag{4·7}$$

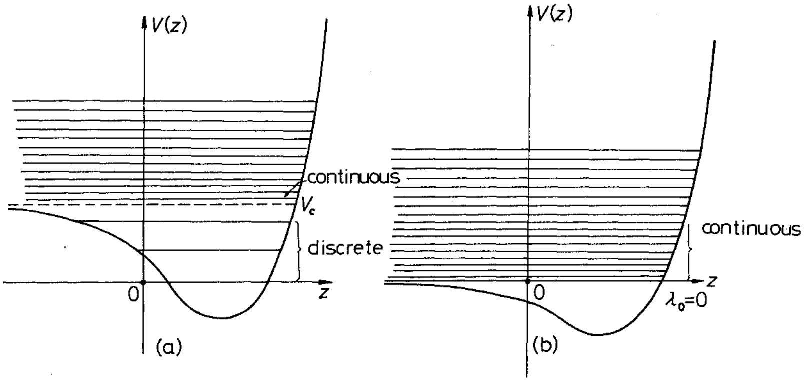

Fig. 1. Effective potential $V(z)$ for $l=1$, and $n=1$ and distribution of spectra, (a) $\gamma>0$, $V_c=\gamma^2/(4\varepsilon)$, (b) $\gamma\to 0$.

for $n>1$, where $L=l-1$, $M=m-1$ and $N=n-1$. From the asymptotic behavior of these potentials, it is easily found that the continuous spectra appear up to $\lambda=0$ (i.e., $\lambda_0=0$) at the critical point, and consequently the long-time tail appears in the cases discussed in § 3.

§ 5. Summary and discussion

In the present paper, we have discussed some general aspects of critical slowing down in multiplicative stochastic processes on the basis of the dynamical scaling law. We have classified the critical slowing down into the three categories, (i) deterministic case $m<2n-1$, (ii) marginal case $m=2n-1$ and (iii) noise-induced case. In the noise-induced case, we have found the interesting result that there appears a situation in which the long-time exponent ψ_p of the moment $\langle x^p(t)\rangle$ is constant in some range of p. The physical mechanism of this noise-induced long-time tail has been found. It is caused by the balance of the nonlinear effect and the noise term. We have evaluated the long-time tail exponent explicitly in general situations (1·5) or (1·11) under certain conditions which have been confirmed on the basis of our previous general criterion of slowing down and also using the distribution of eigenvalues of the corresponding Schrödinger-type equation.

Some exactly soluble models have been discussed to explain the long-time tail explicitly.

It will be desirable to perform the Monte Carlo simulation for the models (1·5) or (1·11), and also to make experiments corresponding to the present nonlinear Langevin equations, for example, by using nonlinear electric circuits.

The present arguments will be easily extended to multicomponent systems such as the Brusselator,[20] and to the periodic spinodal decomposition.[21]

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Appendix A

—— Long-Time Tail in the SKS-Model ——

In the SKS-model (3·13), the origin $x=0$ is a natural boundary but $x=\infty$ is a regular boundary, because the latter can be reached in finite time. This is easily seen not only by examining the Keilson conditions,[22] but also by rewriting (3·13) as

$$\frac{dy}{dt}=g-(m-1)\gamma y+\eta(t) \qquad (\mathrm{A}\cdot 1)$$

in terms of the transformation $y=x^{-(m-1)}/(m-1)$ and by noting that Eq. (A·1) has no singularity at $y=0$.

The boundary condition for $x=\infty$ should be determined by physical considerations. In most cases the reflecting wall is more physical than the absorbing one, because in a physical problem the function $f(x)$ in (1·10) usually has higher order terms, for which $x=\infty$ becomes inaccessible (cf. Appendix B.). Hereafter a reflecting wall is assumed to exist at $x=\infty$ (accordingly at $y=0$).

The Fokker-Planck equation corresponding to (A·1) is given by

$$\frac{\partial}{\partial t}P(y,t)=\frac{\partial}{\partial y}\{(m-1)\gamma y-g\}P(y,t)+\varepsilon\frac{\partial^2}{\partial y^2}P(y,t). \qquad (\mathrm{A}\cdot 2)$$

The stationary distribution function is obtained easily as

$$P_{\mathrm{st}}(y)=\left[\sqrt{\frac{(m-1)\gamma}{2\varepsilon}}\Big/\left\{\frac{\sqrt{\pi}}{2}+\mathrm{Erf}\left(\sqrt{\frac{g^2}{2\varepsilon\gamma(m-1)}}\right)\right\}\right]$$
$$\times\exp\left\{-\frac{(m-1)\gamma}{2\varepsilon}\left(y-\frac{g}{(m-1)\gamma}\right)^2\right\}, \qquad (\mathrm{A}\cdot 3)$$

where $\mathrm{Erf}(x)=\int_0^x e^{-u^2}du$, and consequently the moment $\langle x^p\rangle_{\mathrm{st}}$ in the stationary state is expressed by

$$\langle x^p\rangle_{\mathrm{st}}=(m-1)^{-p/(m-1)}\int_0^\infty y^{-p/(m-1)}P_{\mathrm{st}}(y)dy\,. \qquad (\mathrm{A}\cdot 4)$$

Here (A·3) is valid for $\gamma>0$ and the integral on r.h.s. of (A·4) converges only when $p<m-1$. When γ is small, the moment $\langle x^p\rangle_{st}$ takes the following asymptotic form $\langle x^p\rangle_{st}\sim(\gamma/g)^{p/(m-1)}$, being derived easily. On the other hand, when $\gamma=0$, the time-dependent solution of (A·2) is available, namely we have

$$P(y,t)=\frac{1}{\sqrt{4\pi\varepsilon t}}\sum_{\sigma=\pm1}\exp\left\{-\frac{(1-\sigma)gy_0}{2\varepsilon}-\frac{(y-\sigma y_0-gt)^2}{4\varepsilon t}\right\}$$

$$-\frac{ge^{gy/\varepsilon}}{\varepsilon\sqrt{\pi}}\,\mathrm{Erfc}\left(\frac{y+y_0+gt}{\sqrt{4\varepsilon t}}\right), \tag{A·5}$$

where $\mathrm{Erfc}(x)=\int_x^\infty e^{-u^2}du$. Then, the asymptotic form of $\langle x^p(t)\rangle$ for large t is given by $\langle x^p(t)\rangle\sim\{(m-1)gt\}^{-p/(m-1)}$, using (A·5). Thus we arrive at (3·14).

Appendix B

—— Effect of Higher Order Terms ——

In the case $m<2n-1$ in (3·1), the moments $\langle x^p\rangle$ cannot exist for $p\geq n-1$, namely they are divergent. In the marginal case $m=2n-1$, a similar behavior appears when $p\geq n+g/\varepsilon-1$ as is seen in Appendix C. If higher order terms are present in $f(x)$ and $g(x)$ in (1·10), however, those moments may exist. As Schenzle and Brand[7] studied, the condition that all the positive moments should exist is given by the inequality $F>2G-1$, where F and G are determined by the following relations: $f(x)=O(x^F)$ and $g(x)=O(x^G)$ as $x\to\infty$.

To study long-time tails for this case, we consider the following equation:

$$\frac{dx}{dt}=\gamma x-gx^m-g'x^{2n-1}-g''x^k+x^n\eta(t), \tag{B·1}$$

where $m<2n-1$, $k>2n-1$, $n>1$, $g>0$ and g' and g'' are assumed to be positive. For simplicity $g(x)$ is set as a single power, but the following discussion can be directly extended to other cases, and the conclusion does not change.

By the presence of the fourth term in the r.h.s. of (B·1), both $x=0$ and $x=\infty$ become natural boundaries. The stationary solution of the Fokker-Planck equation corresponding to (B·1) is given by

$$P_{st}(x)=N_0x^{-n-g'/\varepsilon}\exp\{-bx^{-2(n-1)}+ax^{-(2n-1-m)}-cx^{k-2n+1}\}, \tag{B·2}$$

where N_0 is a normalization constant and $a=g/(2n-1-m)\varepsilon$, $b=\gamma/2(n-1)\varepsilon$ and $c=g''/(k-2n+1)$. When $\gamma>0$, stationary moments $\langle x^p\rangle_{st}$ are finite for all p's and are expressed by $\langle x^p\rangle_{st}=F(p)/F(0)$ with

$$F(p)\equiv\int_0^\infty x^{p-n-g'/\varepsilon}\exp\{-bx^{-2(n-1)}+ax^{-(2n-1-m)}-cx^{k-2n+1}\}dx\,. \tag{B·3}$$

According to the dynamic scaling theory in § 2 we need only to find the

b-dependence of $\langle x^p\rangle_{st}$ in order to know the critical slowing down of the moment, because Δ is supposed not to change through the higher order terms.

We examine first the case $g>0$. We use hereafter the notations $K\equiv k-1$, $M\equiv m-1$, $N\equiv n-1$ and $G'\equiv g'/\varepsilon$. The quantity $F(p)$ diverges due to the contribution of the neighborhood of $x=0$ as $b\to 0$. Therefore the term $-cx^{k-2n+1}$ is irrelevant to the asymptotic evaluation. It has only a role of a convergent factor. Using the transformation $x=b^{1/M}y$, (B·3) can be evaluated as follows:

$$F(p)=b^{(p-N-G')/M}\int_0^\infty y^{p-n-G'}\exp[-b^{-(2N-M)/M}y^{-2N}(1-ay^M)-cb^{(K-2N)/M}y^{K-2N}]dy$$

$$\sim\sqrt{\frac{\pi}{NM}}\,b^{(p-G'-1/2)/M}\left(\frac{2N}{a(2N-M)}\right)^{(p-G')/M}\exp\left[\frac{Mb^{-(2N-M)/M}}{2N-M}\left(\frac{a(2N-M)}{2N}\right)^{2N/M}\right]. \tag{B·4}$$

Using that $a\propto g$ and $b\propto\gamma$, we arrive at the result $\langle x^p\rangle_{st}\sim(\gamma/g)^{p/(m-1)}$. It should be remarked that the term $-g'x^{2n-1}$ in (B·2) plays no important role, and that φ_p is essentially determined by (3·1). From the dynamic scaling theory, the long-time exponent ψ_p is given by $p/(m-1)$ for all p's. Thus, the restriction that $p<n-1$ can be removed, if we include higher order terms in (3·1).

Next we consider the case $g=0$ which is a marginal example. If $p<N+G'$, higher order terms are irrelevant, and the asymptotic form of stationary moments for small γ coincides with the exact form (C·7) in Appendix C. If $p>N+G'$, $F(p)$ is convergent even if $b=0$, namely we have

$$\lim_{b\to 0}F(p)=\frac{1}{K-2N}c^{-(p-G'-N)/(K-2N)}\Gamma\left(\frac{p-N-G'}{K-2N}\right). \tag{B·5}$$

Thus the asymptotic form of $\langle x^p\rangle_{st}$ is expressed by

$$\langle x^p\rangle_{st}\sim\frac{2N}{K-2N}b^{(N+G')/2N}c^{-(p-G'-N)/(K-2N)}\Gamma\left(\frac{p-N+G'}{K-2N}\right)\Big/\Gamma\left(\frac{N+G'}{2N}\right). \tag{B·6}$$

The absolute value of (B·6) depends on K but the power of b does not depend on it. This means that higher order terms do not affect the long-time exponent. From (C·7) and (B·6), we obtain the long-time tails as $\langle x^p(t)\rangle\sim t^{-p/2(n-1)}$ for $p<n-1+g'/\varepsilon$ and $\langle x^p(t)\rangle\sim t^{-(n-1+g'/\varepsilon)/2(n-1)}$ for $p>n-1+g'/\varepsilon$ at $\gamma=0$, using the dynamic scaling theory. Therefore the quantity $n-1+g'/\varepsilon$ is equal to the critical value p_c in this case.

In a noise-induced case $m>2n-1$, higher order terms in $f(x)$ have no essential effect.

Appendix C

— *Exact Solution of the Marginal Equation (3·17)* —

The marginal equation (3·17) is transformed by the variable $y=x^{-2(n-1)}$ into the following equation:

$$\frac{dy}{dt}=2(n-1)g-2(n-1)\gamma y-2(n-1)\sqrt{y}\,\eta(t). \tag{C·1}$$

The corresponding Fokker-Planck equation takes the form

$$\frac{\partial}{\partial t}P(y,t)=-\frac{\partial}{\partial y}(a-by)P(y,t)+\varepsilon'\frac{\partial^2}{\partial y^2}yP(y,t), \tag{C·2}$$

where $a=2(n-1)\{g+(n-1)\varepsilon\}$, $b=2(n-1)\gamma$ and $\varepsilon'=4(n-1)^2\varepsilon$. Using the Laplace transformation $Q(k,t)=\mathcal{L}(P(y,t))$ we obtain the following partial differential equation of first order:

$$\frac{\partial}{\partial t}Q(k,t)=-akQ(k,t)-k(b+\varepsilon' k)\frac{\partial}{\partial k}Q(k,t). \tag{C·3}$$

This equation can be solved by the usual method of characteristics. The solution of (C·3) with the initial condition $P(y,0)=\delta(y-y_0)$ is given by

$$Q(k,t)=\left\{1+\frac{\varepsilon'}{b}k(1-e^{-bt})\right\}^{-a/\varepsilon'}\exp\left\{-\frac{y_0 k e^{-bt}}{1+b^{-1}\varepsilon' k(1-e^{-bt})}\right\}. \tag{C·4}$$

The inverse Laplace transformation of (C·4) yields

$$P(y,t)=\frac{b}{\varepsilon'}\frac{1}{1-e^{-bt}}\left(\frac{ye^{bt}}{y_0}\right)^{a/2\varepsilon'-1/2}$$
$$\times\exp\left\{-\frac{b}{\varepsilon'}\frac{y+y_0e^{-bt}}{1-e^{-bt}}\right\}I_{a/\varepsilon'-1}\left(\frac{2be^{-bt/2}(y_0y)^{1/2}}{\varepsilon'(1-e^{-bt})}\right). \tag{C·5}$$

This result was obtained first by Wong.[25] The moment $\langle x^p(t)\rangle$ can be represented in terms of confluent hypergeometric function as

$$\langle x^p(t)\rangle=\left[\frac{\gamma}{2(n-1)\varepsilon}\frac{1}{1-e^{-2(n-1)\gamma t}}\right]^{p/2(n-1)}$$
$$\times\frac{\Gamma\left(\frac{g}{2(n-1)\varepsilon}+\frac{1}{2}-\frac{p}{2(n-1)}\right)}{\Gamma\left(\frac{g}{2(n-1)\varepsilon}+\frac{1}{2}\right)}$$
$$\times{}_1F_1\left(\frac{p}{2(n-1)};\frac{g}{2(n-1)\varepsilon}+\frac{1}{2};-\frac{\gamma x_0^{-2(n-1)}}{2(n-1)\varepsilon(e^{2(n-1)\gamma t}-1)}\right) \tag{C·6}$$

for $p<n-1+g/\varepsilon$, where x_0 is an initial value. When $p\geq n-1+g/\varepsilon$, the moments

$\langle x^p(t)\rangle$ diverge.

Stationary moments and long-time tails are obtained from (C·6) for γ fixed and $t\to\infty$, and for t fixed and $\gamma\to 0$, respectively. That is, we have

$$\langle x^p\rangle_{\rm st}=\left[\frac{\gamma}{2(n-1)\varepsilon}\right]^{p/2(n-1)}\frac{\Gamma\left(\dfrac{g}{2(n-1)\varepsilon}+\dfrac{1}{2}-\dfrac{p}{2(n-1)}\right)}{\Gamma\left(\dfrac{g}{2(n-1)\varepsilon}+\dfrac{1}{2}\right)} \tag{C·7}$$

for $\gamma>0$, and

$$\langle x^p(t)\rangle=\left[\frac{1}{4(n-1)^2\varepsilon t}\right]^{p/2(n-1)}\frac{\Gamma\left(\dfrac{g}{2(n-1)\varepsilon}+\dfrac{1}{2}-\dfrac{p}{2(n-1)}\right)}{\Gamma\left(\dfrac{g}{2(n-1)\varepsilon}+\dfrac{1}{2}\right)}$$
$$\times{}_1F_1\left(\frac{p}{2(n-1)};\frac{g}{2(n-1)\varepsilon}+\frac{1}{2};-\frac{x_0^{-2(n-1)}}{4(n-1)^2\varepsilon t}\right)$$
$$\sim t^{-p/2(n-1)},\quad (t\to\infty) \tag{C·8}$$

at $\gamma=0$. Thus we arrive finally at (3·22)~(3·24).

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