SPATIAL PERIOD-DOUBLING IN OPEN FLOW

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Coupled logistic lattices with asymmetric coupling in space, with a fixed boundary condition at the left end, are investigated. The system shows a period-doubling bifurcation to chaos as a lattice point goes downflow. In contrast with usual period-doubling in low-dimensional systems, (i) no scaling behavior has been found, (ii) low noise is important for the bifurcation structures. The system corresponds to a model for an open flow, which may be of use for the study of the onset of turbulence in pipe flows.

1. Introduction and motivation. Recent studies of chaos [1,2] have made clear some aspects of turbulence, especially those in closed systems such as Bénard convection or Taylor vortices. On the other hand, studies of open flow systems from the viewpoint of dynamical systems are rare [2,3]. One interesting feature for an open flow system lies in the change of a structure of a flow as it goes downstream [4], such as the growth of a disturbance or the development of Karman eddies. In that sense, "space" can be considered as a kind of bifurcation parameter for an open flow system. In the present letter a coupled map lattice (CML) model is used to consider such flow systems.

A CML is a system in which a set of low-dimensional mappings are coupled on a lattice [‡1] [5,6]. Here, a coupled logistic lattice with an asymmetric coupling is considered, i.e.,

$$x_{n+1}(i) = f(x_n(i)) + \epsilon[\alpha f(x_n(i+1)) + (1-\alpha)f(x_n(i-1)) - f(x_n(i))], \qquad (1)$$

where $i = 1, 2, ..., N$ is a lattice site and $f(x) = 1 - ax^2$. In the present letter we present some results for the case with one-way coupling ($\alpha = 0$) and with the boundary condition $x(1)$ = fixed at the unstable fixed point of a single logistic map, $x^* = (\sqrt{1+4a} - 1)/2a$, though essential features do not change if the coupling is asymmetric with the boundary condition $x(1)$ = fixed and $x(N) = x(N-1)$.

2. Spatial period doubling. As the lattice site is increased, spatial period doubling is observed for various parameters. The pattern may be described as follows: At lattice sites $i(1) < i < i(2)$, $x_n(i)$ is a period-two cycle (values of the $x(i)$ at that cycle can differ by lattice sites), at $i(2) < i < i(4)$, $x_n(i)$ is a period-four cycle and so on. At some parameters (a, ϵ), the spatial period doubling stops at some order 2^k and the system settles down to a cycle with the period 2^k for $i > i(k-1)$. For other parameters, the time series of $x_n(i)$ shows a chaotic behavior after a finite number of spatial period doublings usually nonlinearity a or a smaller coupling ϵ gives a chaotic behavior). Once chaos is attained at some lattice points i_c, no periodic behavior reappears for $i > i_c$. Some examples are shown in figs. 1a–d, where the initial conditions are chosen to be $x(i) = x^*$ + small disturbance, though the patterns do not depend on the initial conditions so much. The following points should be noted.

(a) The period doubling is not caused by the change of a bifurcation parameter. The bifurcation occurs

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[‡1] For other studies of CMLs see ref. [7].

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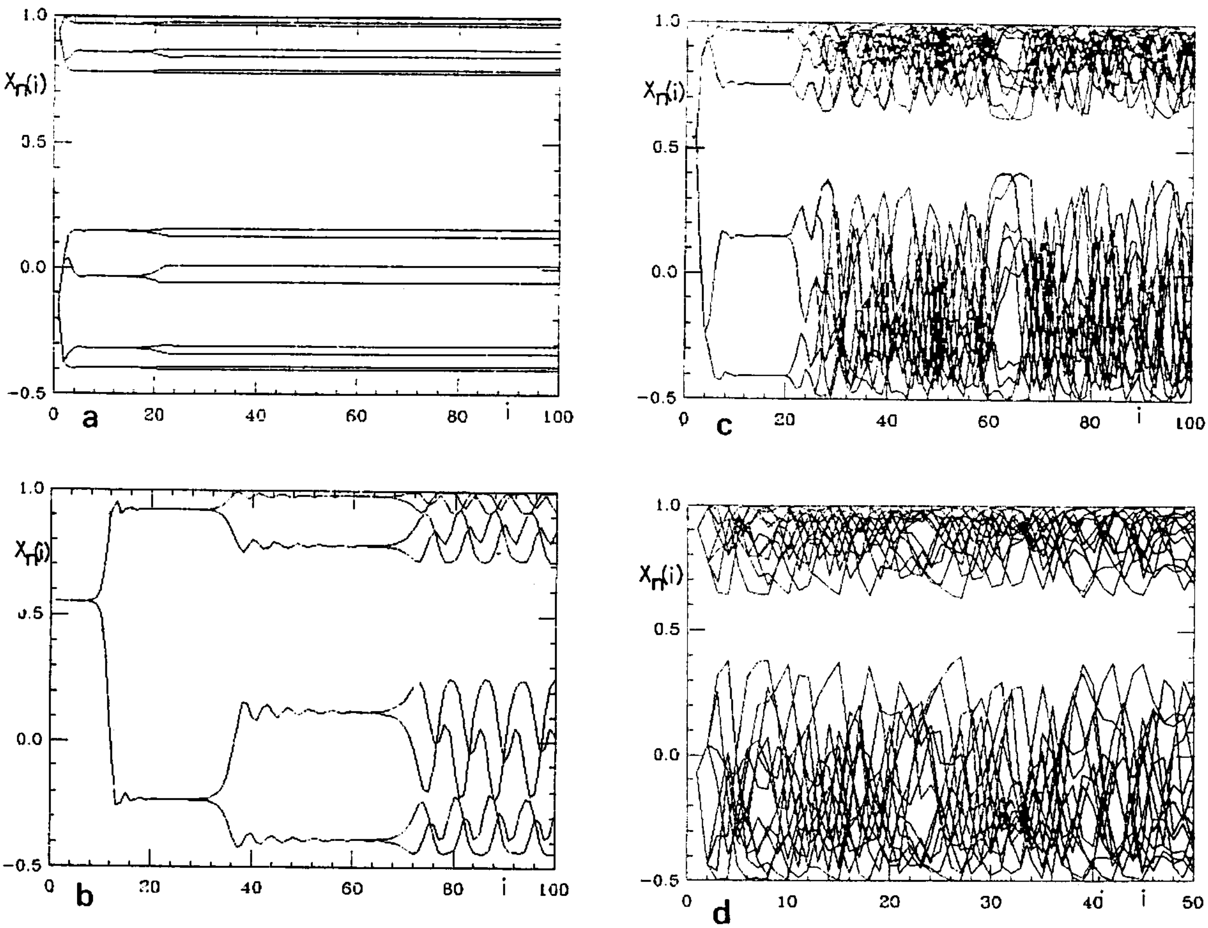
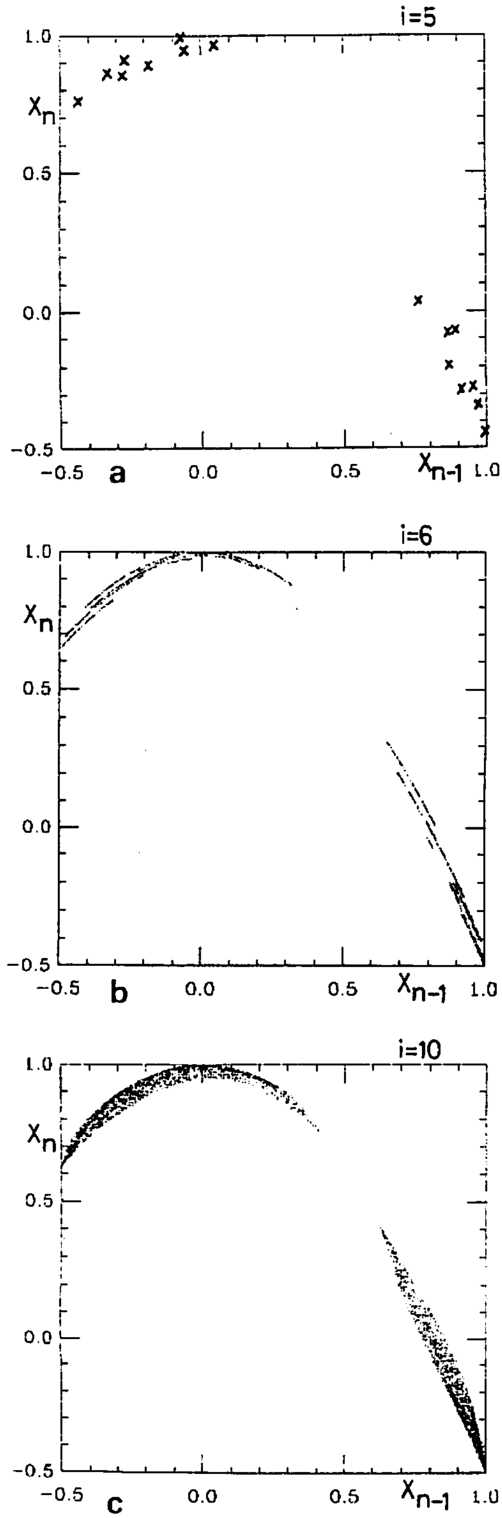


Fig. 1. Patterns by eq. (1). The $x_n(i)$ are plotted for n = 5001, 5002, ..., and 5020. Initial conditions are $x(i) = x^* + 0.01$, though the pattern does not change so much if different initial conditions are taken. The piecewise lines denote $(i, x_n(i))$ – $(i + 1, x_n(i + 1))$. The system size N is 100 for (a)–(c) and 50 for (d). (a) a = 1.4 and ϵ = 0.1. (b) a = 1.45 and ϵ = 0.5. (c) a = 1.5 and ϵ = 0.3. (d) a = 1.5 and ϵ = 0.2 (for n = 5001, 5002, ..., and 5025).

automatically as a lattice site goes downflow. In that sense, "space" plays the role of bifurcation parameter.

(b) No scaling relations are observed. For example, let us consider a lattice point $i(k)$, at which the doubling from 2^{k-1} to 2^k occurs. The points $i(k)$ do not accumulate as k becomes large. On the contrary, the distance between $i(k + 1)$ and $i(k)$ becomes larger and larger as k increases. Scaling in a parameter space a or ϵ are not observed either, which is quite different from the common sense in a low-dimensional chaos theory [8]. A possible reason for this is discussed in section 5.

3. Mechanism. The mechanism of the above phenomenon seems to be rather simple. Assume that $x_n(i)$ is a cycle with period 2^k. Then the mapping at the site $i + 1$ is a logistic map modulated by the period 2^k. Then the amplitude of oscillation at the site $i + 1$ can become larger than that at the site i or a pitchfork bifurcation from 2^k to 2^{k+1} cycle occurs. In figs. 2a–c, $x_{n+1}(i)$ versus $x_n(i)$ are plotted for i = 5, 6, and 10. At these parameters in the figures, transition to chaos from 16-cycle occurs at the site i = 6, and the stochastic motion is propagated to the downstream.



◀ Fig. 2. $(x_{n-1}(i), x_n(i))$ is plotted for n = 5001, 5002, ..., and 7000 from eq. (1) with the same initial condition as for fig. 1. Lattice sites are i = 5 ((a); period-16), i = 6 (b) and i = 10 (c).

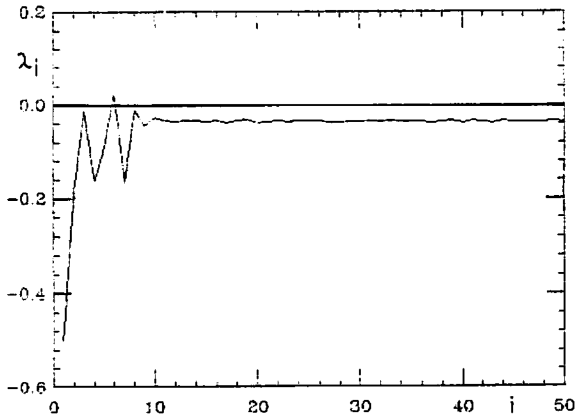


Fig. 3. Lyapunov exponent as a function of lattice site. The parameters and the initial condition are the same as in fig. 2. Calculations are performed by 10^5 iterations after 5000 times transients are dropped.

4. Lyapunov exponents and convective instability. In a system with a one-way coupling and the fixed boundary condition for the left end, the jacobian for the CML (1) is a triangular matrix. Thus an eigenvalue of the product of jacobians is given by the product of each diagonal element. Lyapunov exponents for the system (1), therefore, are easily calculated and each exponent has a one-to-one correspondence with the lattice site. The Lyapunov exponent as a function of the lattice site is given in fig. 3, where the parameters are same as for fig. 2. We note that the exponent is positive only at the site 6 and it approaches to some constant (negative) value for the downstream. The stochastic motion at the downflow is essentially due to the propagation of the turbulence of the upstream and cannot be represented by the usual Lyapunov exponents. The convergence of the exponent at the downstream means that the flow approaches some stationary state there.

In connection with the Lyapunov exponents, we have to be careful in the difference between convective instability and absolute instability. If the perturbation grows in a moving frame, it is called "convective instability", while, absolute instability means the instability only in a stationary frame [9]. In many cases our system shows a convective instability.

As the simplest case, let us consider the stability of the homogeneous solution $x(i) = x^*$. From the calculation of the jacobian, it is stable if $-1 < (1-\epsilon) \times f'(x^*) < 1$, i.e., $(1-\epsilon)(\sqrt{1+4a}-1) < 1$. The solution, however, is unstable for some comoving frame if $-1 < f'(x^*) < 1$. Thus, the fixed point is convectively unstable if $(\sqrt{1+4a}-1) > 1$.

5. Importance of a small noise. As has been shown by Deissler for the generalized time dependent Ginzburg–Landau equation, a small noise plays a very important role for the system with convective (or spatial in his terminology) instability [9]. This is also true of our system. In a variety of cases, single and double precision calculations give different results, in the sense that the bifurcation from 2^k to 2^{k+1} occurs at different sites by precisions, if the 2^k-cycle is convectively unstable but not absolutely unstable ($i(k+1)$ is larger for the double precision). One reason that any scaling relations are not found lies in the above sensitive dependence on a small error in our system.

It is of importance to study the system (1) with a small noise added on every site. Numerical simulations of such systems show (see fig. 4) that (i) spatial period doubling occurs in the same manner as in the deterministic case, but (ii) kinks are generated at the upstream, which is due to the phase change of the oscillation at the site where the bifurcation occurs and (iii) the kinks are propagated with a constant speed to the downstream, where the speed is determined by the difference of the phase by a kink while the density of kinks increases as the strength of noise gets larger.

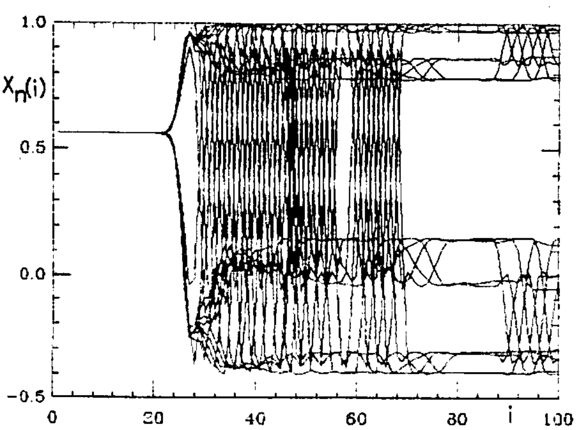


Fig. 4. $x_n(i)$ are plotted for $n = 5001, 5002, ...,$ and 5020 for eq. (1) with a noise homogeneously distributed in the interval $(-5 \times 10^{-14}, 5 \times 10^{-14})$. $a = 1.4$ and $\epsilon = 0.5$, with the same initial condition as in fig. 1. Note the existence of kinks.

6. Discussion. In the present letter, we presented the results for the one-way coupling and the boundary condition fixed at the left end to $x(0) = x^*$. The main results, however, do not change if the coupling is asymmetric and the boundary condition is fixed at some value at the left and free at the right end. The reason that we take the one-way coupling for the illustration of the period doubling in space is that it is the simplest model and the results are less complicated.

Studies of open flow systems from the viewpoints of dynamical systems have just been started and a lot of problems are left for future, such as how the dimension changes as the flows go downstream, how the information or perturbation propagates to downstream, and the statistical property of the turbulence at the downstream.

It is not sure whether our phenomenon can be observed in open flow experiments or in a numerical simulation of the partial differential equations which include the term of first spatial derivatives. It will be of interest to search for the spatial bifurcations in open flow systems. Also, it is desirable to make experiments in open fluid systems such as to take a Poincaré map or to calculate the dimension of a time series or to measure the power spectra at various points of the flow.

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References

[1] Y. Kuramoto, ed., Chaos and statistical methods (Springer, Berlin, 1984);
K. Kaneko, Collapse of tori and genesis of chaos in dissipative systems (World Scientific, Singapore), to be published.
[2] T. Tasumi, ed., Turbulence and chaotic phenomena in fluids (North-Holland, Amsterdam, 1984).

[3] K. Sreenivasan, preprint Yale University (1984).
[4] R. Feynman, R.B. Leighton and M. Sands, Feynman lectures on physics, Vol. 2 (Addison-Wesley, Reading, 1964).
[5] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480; in: Dynamical problems in soliton systems, ed. S. Takeno (Springer, Berlin, 1985) pp. 272–277; Turbulence in coupled map lattices, to be published in Physica D; Spatiotemporal Intermittency in CML, preprint: Lyapunov spectra and vectors in CML, in preparation.
[6] R.J. Deissler, Phys. Lett. 100A (1984) 451.
[7] J. Crutchfield, private communication;
J.D. Keeler and J.D. Farmer, private communication;
R. Kapral, Phys. Rev. A31 (1985) 3868;
Y. Aizawa, Prog. Theor. Phys. 72 (1984) 662;
Y. Aizawa and I. Nishikawa, private communication;
T. Yamada and H. Fujisaka, preprint.
[8] M.J. Feigenbaum, J. Stat. Phys. 19 (1978) 25; 21 (1979) 669.
[9] R.J. Deissler, preprint, J. Stat. Phys., to be published.
[10] R.J. Deissler and K. Kaneko, preprint Los Alamos (1985).