VELOCITY-DEPENDENT LYAPUNOV EXPONENTS AS A MEASURE OF CHAOS FOR OPEN-FLOW SYSTEMS

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Many flows in nature are "open flows" (e.g. pipe flow). We study two open-flow systems driven by low-level external noise: the time-dependent generalized Ginzburg–Landau equation and a system of coupled logistic maps. We find that a flow which gives every appearance of being chaotic may nonetheless have no positive Lyapunov exponents. By generalizing the notions of convective instability and Lyapunov exponents we define a measure of chaos for these flows.

1. Introduction

If a dynamical system is such that two nearby trajectories in phase space diverge exponentially on the average, the system is usually defined as being chaotic. In other words the system has at least one positive Lyapunov exponent [1–6]. Fluid systems for which this definition can be applied without any problems – at least in principle – are the so-called "closed-flow" systems, examples of which are Taylor–Couette flow and Rayleigh–Bénard convection. However, as we shall see, problems can result in applying this definition to systems which have a mean flow velocity such as "open-flow" systems, examples of which are fluid flow in a pipe and fluid flow over a flat plate [‡1].

In this paper we consider two open-flow systems which are driven by low levels of external noise: the time-dependent generalized Ginzburg–Landau equation (in the stationary frame of reference) and a system of coupled logistic maps. Since perturbations may grow or decay depending on the frame of reference [8,9], one may expect problems in applying the usual definition of Lyapunov exponents to these systems; and indeed we find that a system which gives every appearance of being chaotic may nonetheless have no positive Lyapunov exponents. Considering that many of the flows in nature are open flows, it is important to have a measure of chaos for such flows. By generalizing the concept of convective instability [8,10–12] (to include perturbations about a general state) and by generalizing the definition of Lyapunov exponents, we define a measure of chaos for these flows [‡2].

2. The Ginzburg–Landau equation

The time-dependent generalized Ginzburg–Landau equation in the stationary frame of reference is

$$\frac{\partial\psi}{\partial t}=a\psi-v_g\frac{\partial\psi}{\partial x}+b\frac{\partial^2\psi}{\partial x^2}-c|\psi|^2\psi, \tag{1}$$

where the dependent variable $\psi(x, t)$ is in general complex; a, b and c are constants which are in general complex, and v_g is the group velocity. Real and imaginary parts are subscripted with r and i respectively. The term with the first order spatial derivative is a convective term and is responsible for the "mean flow". The boundary conditions are $\psi=0$ at

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[‡1] For some discussion on "closed flow" and "open flow" fluid systems see ref. [7].
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the left boundary and $d^2\psi/dx^2=0$ at the right boundary to approximate an open boundary. (This approximates an open boundary since the value of ψ at the boundary point is given by a linear extrapolation from the values of ψ at two inner grid points [14].) Eq. (1) has been studied for the case $v_g=0$ [15–19] and for the convectively (i.e. spatially) unstable case where v_g is greater than a critical value ($v_g>2|b|(a_r/b_r)^{1/2}$) [8]. Two fluid systems with nonzero group velocity to which eq. (1) is related are plane Poisseulle flow [20] and wind-induced water waves [21]. It is also related to many other systems [15,22–24].

Under conditions when the equilibrium state is convectively unstable, eq. (1) is numerically solved in the presence of low level external noise starting with the initial condition $\psi(x,0)=0$. (A convective instability is one in which a small localized perturbation moves spatially such that the perturbation grows only in a moving frame of reference, eventually damping at any given stationary point [8,10–12].) Second order Runge–Kutta is used in the time differencing (with $\Delta t=0.01$) and fourth order differencing is used in the space differencing (with $\Delta x=0.3$) except at the grid points adjacent to the boundaries where second order differencing is used. Noise is introduced into the system by adding, at each time step, random numbers uniformly distributed between $-r$ and r to ψ_r and ψ_i at all grid points except the boundary points. This amounts to adding a noise term $\eta(x,t)$ to eq. (1). If noise is added at only the left boundary the final results will be the same. Cray single precision (14 digit accuracy) is used in the calculations.

Fig. 1 shows a plot of ψ_r as a function of x for a particular t after the system has reached a statistically steady state. The noise near the left boundary is spatially and selectively amplified resulting in the formation of spatially growing waves. At some spatial point the waves become macroscopic and nonlinear effects become important causing the waves to saturate (assuming $c_r>0$) producing a structure. The nonlinear dynamics causes the structure to change in a chaotic fashion with time. This may be seen in fig. 2 where ψ_r is plotted as a function at t at $x=210$. For $x>150$ the flow appears to be fully developed. For more details the reader is referred to ref. [8]. (The fact that the irregular behavior is generated mainly by the nonlinear dynamics may be seen by imposing periodic boundary conditions instead and by removing the noise. Under these conditions the qualitative behavior of the periodic system is essentially identical to that of the fully developed region of the original system.)

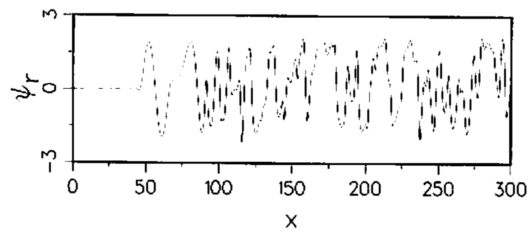

Fig. 1. Plot of ψ_r (in Ginzburg–Landau equation, eq. (1)) as a function of x for a given t ($t=500$) after transients have settled down. $a=2$, $v=6$, $b_r=1$, $b_i=-1$, $c_r=0.5$, $c_i=1$. Noise level $=r=10^{-7}$. The microscopic noise near the left boundary grows spatially to macroscopic proportion resulting in the observed structure. The usual largest Lyapunov exponent for this flow is negative even though the flow appears to be very chaotic (see fig. 2).

Let $\delta\psi(x,t)$ be an infinitesimal perturbation about the state $\psi(x,t)$. This perturbation satisfies the following equation:

$$\frac{\partial\delta\psi}{\partial t}=a\delta\psi-v_g\frac{\partial\delta\psi}{\partial x}+b\frac{\partial^2\delta\psi}{\partial x^2}-2c|\psi|^2\delta\psi-c\psi^2\delta\psi^*. \tag{2}$$

Consider an initial perturbation $\delta\psi(x,0)$ localized in the region $\{x_1,x_2\}$. We may define a *velocity-dependent* Lyapunov exponent (the largest one) as follows:

$$\lambda(v;x_1,x_2)=\lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{\zeta(v,x_1,x_2,t)}{\zeta(v,x_1,x_2,0)}\right), \tag{3}$$

where

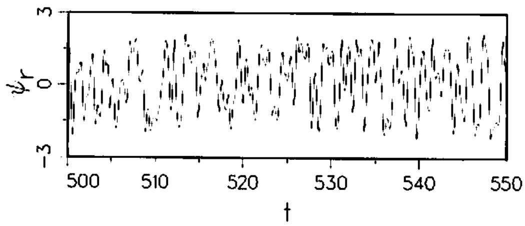

Fig. 2. Plot of ψ_r as a function of t at $x=210$ (see fig. 1). $a=2$, $v=6$, $b_r=1$, $b_i=-1$, $c_r=0.5$, $c_i=1$.

$$\zeta(v, x_1, x_2, t) = \left(\int_{x_1+vt}^{x_2+vt} |\delta\psi(x, t)|^2 \, dx \right)^{1/2} .$$

Here v refers to the velocity of the frame of reference from which the system is observed. For $v>0$ the definition assumes that the system is unbounded in the $+x$ direction. For $v=0$ and for x_1 and x_2 corresponding to the boundaries of the system this definition reduces to the usual definition for the largest Lyapunov exponent. Let g_m be that v which gives the maximum value of $\lambda(v;\, x_1,\, x_2)$. If $\lambda(v_m;\, x_1,\, x_2)>0$, we say that the system is chaotic in the region $\{x_1+v_m t,\, x_2+v_m t\}$ and that $\lambda(v_m;\, x_1,\, x_2)$ is a measure of that chaos [3]. The term chaos is appropriate here since the separation of trajectories – in *some* frame of reference – is generated by the deterministic dynamics. Since the concept of convective instability is usually associated with perturbations about a stationary state, eq. (3) generalizes this concept by considering perturbations about a general state.

If the usual definition for the largest Lyapunov exponent is applied to the system of figs. 1 and 2 we find that the exponent is negative (i.e. $\lambda(0;\, 0,\, 300) = -2.55 \pm 0.02$) even though the flow seen in figs. 1 and 2 appears to be chaotic. The behavior described here occurs for a wide range of parameter values and is not restricted to the parameter values of figs. 1 and 2. λ was calculated by solving eq. (2) along with eq. (1). We note that noise is added only to eq. (1) and not to eq. (2) since the noise added to eq. (1) is independent of ψ and thus does not contribute a term to eq. (2) when ψ is varied. If the definition (3) is applied to a fully developed portion of the flow ($x_1 = 180$ and $x_2 = 300$) with $v=0$ we get the same value (i.e. $\lambda(0;\, 180,\, 300) = -2.55 \pm 0.02$). The same value occurs since, for large t, $\delta\psi$ is significant only in a region near the right boundary. The reason that these values are negative is that, even though the perturbation $\delta\psi$ may be initially growing, the perturbation is moving at a sufficiently large velocity so that both edges of the perturbation are moving in the same direction, allowing $\delta\psi$ behind the perturbation to approach zero (i.e. $\lim_{t\to\infty}\delta\psi(x,\, t)\to 0$ for an arbitrary fixed value of x in the interval $\{x_1,\, x_2\}$). Thus the bulk of the perturbation eventually moves out through the right boundary leaving only its trailing edge which decreases with time.

As v is increased from zero, $\psi(v;\, x_1,\, x_2)$ will increase until it becomes positive, eventually reaching a maximum (corresponding to the region $\{x_1+vt,\, x_2+vt\}$ moving on the average with the growing perturbation), and then decrease until it again becomes negative (corresponding to the region $\{x_1+vt,\, x_2+vt\}$ "outrunning" the perturbation). An obvious difficulty in directly applying definition (3) with $v>0$ is that the system must be very long in order to get an accurate value for λ since the region $\{x_1+vt,\, x_2+vt\}$ wil eventually reach the right boundary of the system. To circumvent this difficulty we transform eqs. (1) and (2) for the region $\{x_1+v_g t,\, x_2+v_g t\}$ to a frame of reference moving at $v=v_g$ (from symmetry the maximum value of λ will occur at $v=v_g$). This reduces to solving eqs. (1) and (2) without the convective term. We take boundary conditions to approximate open boundaries at both boundaries of this region (i.e. $d^2\psi/dx^2=0$) and take x_2-x_1 sufficiently large so that the boundary conditions will have an insignificant effect on the value of ψ and so that ψ is independent of x_1 and x_2. Here we took $x_2-x_1=180$. We then find $\lambda(v_g)=0.466\pm0.004$, where we have dropped the dependence on x_1 and x_2. This value is also independent of noise level for sufficiently low noise level. Since λ is positive we conclude that the fully developed portion of the flow in fig. 1 is chaotic and that $\lambda=0.466\pm0.004$ is a measure of that chaos.

Since experimentalists many times have only single time series at their disposal, an important question is whether velocity-dependent Lyapunov exponents can be calculated from reconstructions of single time series [14]. Exponents resulting from a reconstruction of a single time series in the stationary (i.e. laboratory) frame of reference for an open-flow fluid system will clearly not give reasonable val-

[3] The definition (3) should only be applied to fully developed flows. For example, it would not be applied to convectively unstable stationary states or convectively unstable periodic (in time and space) states just as the usual definition of Lyapunov exponents is not applied to unstable fixed points or unstable periodic orbits. Also (although here we are interested only in the largest exponent) the other velocity-dependent exponents can be calculated in the same fashion as the spectrum of usual Lyapunov exponents.

[4] For algorithms for the calculation of Lyapunov exponents for some physical fluid systems see refs. [6,25].

ues for velocity-dependent Lyapunov exponents. In fact, the largest Lyapunov exponent calculated for such a time series for a turbulent system will be positive (since the time series is aperiodic) even though nearby trajectories may be converging on the average in the stationary frame of reference. (For example, the largest Lyapunov exponent calculated from a reconstruction of the time series of fig. 2 will be positive since the time series is aperiodic, even though the largest velocity-dependent Lyapunov exponent in the stationary frame of reference is negative). This reflects the fact that, on the average, there is a constant flow of new fluid particles through the region of interest. However, in a frame of reference moving with the average velocity of the fluid this will no longer be a problem, and a reconstruction from a time series obtained in the moving frame of reference should give a reasonable value for the maximal value of the largest velocity-dependent Lyapunov exponent and therefore a reasonable value for the measure of the chaos. For example, for fluid flow in a pipe, the laser (assuming measurements are made with laser Doppler techniques) could be mounted on a cart which moves parallel to the pipe at a velocity equal to the average velocity of the fluid.

3. A system of coupled logistic maps

As another example of a chaotic open-flow system we now consider a coupled map lattice system with asymmetric coupling. A coupled map lattice system is a spatially extended dynamical system that is discrete in both time and space [9,26–34]. It is numerically more tractable than a partial differential equation (while still preserving the essential features) and thus allows us to directly calculate the velocity-dependent Lyapunov exponent for a range of velocities. For examples of map lattice systems that are convectively unstable see refs. [9,34].

The coupled map lattice system considered here is:

$$X_{n+1}^{(i)} = (1-d)f(X_n^{(i)}) + d[\alpha f(X_n^{(i-1)}) + (1-\alpha)f(X_n^{(i+1)})] + \epsilon\sigma_n^{(i)}, \quad (4)$$

where $f(X) = 1 - aX^2$ is the logistic map and $i = 1,2,...,N$. n and i are integers representing discrete time and space variables respectively. Here we choose

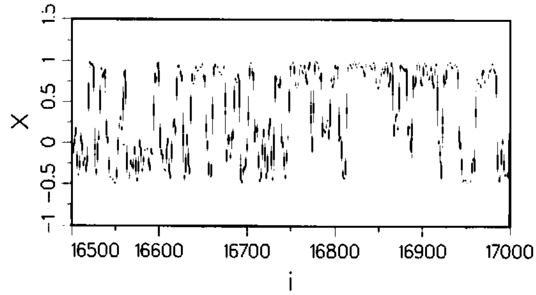

Fig. 3. Plot of $X_n^{(i)}$ (for a portion of the coupled map lattice system, eq. (4)) as a function of i for a given value of n after transients have settled down. $a=1.5$, $d=0.4$, $\alpha=1$. Noise level $=\epsilon=10^{-9}$. The largest velocity-dependent Lyapunov exponent $\lambda(v)$ for this flow is plotted as a function of v in fig. 4.

$\alpha=1$ (one-way coupling) to model a first order derivative. Similar behavior occurs for other values of α as long as the system is sufficiently asymmetric. The boundary condition is chosen to be $X_n^{(0)} = X^*$ where X^* is the unstable fixed point $X^* = (\sqrt{1+4a} - 1)/2a$ of the map $X' = f(X)$. The term $\epsilon\sigma_n^{(i)}$ represents white noise where σ_n is a random number uniformly distributed between -0.5 and 0.5. We let the system evolve from the initial state $X_0^{(i)} = X^*$ under conditions when the system is convectively unstable. As with the Ginzburg–Landau equation, the noise near the left boundary is amplified spatially resulting in a structure. Fig. 3 shows a plot of $X_n^{(i)}$ as a function of i for a portion of the system (far into the fully developed region) for a particular value of n after the system has reached a statistically steady state. This system also undergoes a spatial period doubling in a region to the left of the fully developed flow [34].

Instead of directly using definition (3) appropriately modified for discrete time and space (i.e. $t \to n$, $x \to i$, $vt \to [vn]$, $\delta\psi(x,t) \to \delta X_n^{(i)}$, and $\int \to \Sigma$) we formulate the problem using jacobian matrices. Here $[vn]$ means the integer part of vn, $\delta X_n^{(i)}$ is an infinitesimal perturbation about the state $X_n^{(i)}$, and $0 \leqslant v \leqslant 1$ (i.e. $v = j/k$ where $j = 0,1,...,$ or k). First we calculate the jacobian matrix $J_n = \partial X_{n+1}^{(i'+[v(n+1)])}/\partial X_n^{(j'+[vn])}$ for eq. (4) for $i_1 \leqslant i' \leqslant i_2$ and $i_1 \leqslant j' \leqslant i_2$. This matrix maps $\delta X_n^{(i)}$ for $i_1 + [vn] \leqslant i \leqslant i_2 + [vn]$ into $\delta X_{n+1}^{(i)}$ for $i_1 + [v(n+1)] \leqslant i \leqslant i_2 + [v(n+1)]$. Therefore, as the iteration number n increases from zero, the region $\{i_1 + [vn],\ i_2 + [vn]\}$ will move "downstream" (i.e.

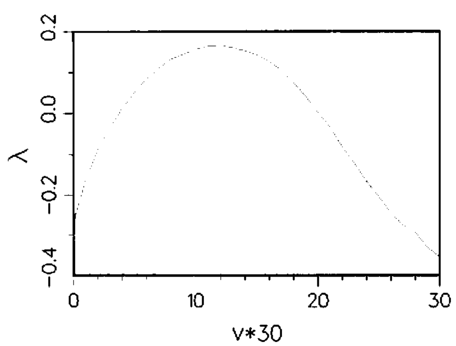

Fig. 4. Plot of the largest velocity-dependent Lyapunov exponent $\lambda(v)$ as a function of v for the flow seen in fig. 3. $i_1=1000$, $i_2=1200$. $v=0/30, 1/30, 2/30, ..., 30/30$. The exponent is seen to be negative for the stationary frame of reference (i.e. $v=0$) and positive for some moving frames of reference.

to larger values of i) with an average velocity of v. We choose i_1 and i_2 such that this region is in the fully developed portion of the flow and take i_2-i_1 sufficiently large so that $\lambda(v)$ is independent of i_1 and i_2. We then have $\lambda(v)=\lim_{n\to\infty}\log\gamma$ where γ is the largest eigenvalue of $J_{n-1}J_{n-2}...J_0$.

Fig. 4 shows a plot of $\lambda(v)$ as a function of v. This behavior is not restricted to the parameter values of fig. 4 but occurs for a wide range of parameter values. Also the plot is insensitive to the noise level for sufficiently low noise level. We see that λ is negative for $v=0$. As v is increased from 0, λ increases until it reaches a positive maximum at $v=d$, and then decreases until it again becomes negative. Since the maximum value of $\lambda=0.1665\pm0.004>0$ we say that the fully developed portion of the flow is chaotic and that $\lambda=0.1665\pm0.004$ is a measure of that chaos.

4. Conclusions

We have studied two open-flow systems that are driven by low levels of external noise – a partial differential equation and a coupled map lattice. We have shown that the usual definition of Lyapunov exponents is inadequate as a measure of the chaotic dynamics for such systems and therefore have generalized the definition of Lyapunov exponents and the concept of convective instability to define a measure of chaos for these systems. The basic source of the inadequacy of the usual definition results from the following: even though two nearby trajectories may exponentially *converge* on the average in the stationary frame of reference, a moving frame of reference may exist in which nearby trajectories exponentially *diverge* on the average.

There are many open-flow fluid systems with a mean flow velocity, such as fluid flow in a pipe, channel flow, and fluid flow over a flat plate. The notion of a velocity-dependent Lyapunov exponent is therefore important in the understanding of the turbulent behavior of such systems. Although this quantity is straightforward (though time consuming) to calculate for numerical solutions of the Navier–Stokes equations, a challenge is to develop a practical algorithm for the calculation of this quantity (for values other than the maximal value) for experimental fluid systems (see footnote 4).

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References

[1] V.L. Oseledec, Moscow Math. Soc. 19 (1968) 197.
[2] G. Benettin and L. Galgani, Phys. Rev. A 14 (1976) 2338.
[3] I. Shimada and T. Nagashima, Prog. Theor. Phys. 61 (1979) 1605.
[4] J.D. Farmer, Physica D 4 (1982) 366.
[5] Y. Pomeau, A. Pumir and P. Pelce, J. Stat. Phys. 37 (1984) 39.
[6] J.P. Eckmann and D. Ruelle, Rev. Mod. Phys. 57 (1985) 617.
[7] K.R. Sreenivasan, in: Frontiers in fluid mechanics, eds. S.H. Davis and J.L. Lumley (Springer, Berlin, 1985).
[8] R.J. Deissler, J. Stat. Phys. 40 (1985) 371; Spatially growing waves, intermittency, and convective chaos in an open-flow system, Los Alamos report LA-UR-85-4211, to be published in Physica D (1987).
[9] R.J. Deissler, Phys. Lett. A 100 (1984) 451.
[10] R.J. Briggs, Electron-stream interaction with plasmas (MIT Press, Cambridge, MA, 1964).
[11] A. Bers, in: Plasma physics, eds. C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1975).
[12] P. Huerre and P.A. Monkewitz, J. Fluid Mech. 159 (1985) 151.

[13] R.J. Deissler and K. Kaneko, in: Dimensions and entropies in chaotic systems, ed. G. Mayer-Kress (Springer, Berlin, 1986).
[14] T. Nitta, J. Met. Soc. Japan 40 (1961) 13.
[15] Y. Kuramoto, Prog. Theor. Phys. 56 (1976) 679; Suppl. Prog. Theor. Phys. 64 (1978) 346.
[16] H.T. Moon, P. Huerre and L.G. Redekopp, Phys. Rev. Lett. 49 (1982) 458; Physica D 7 (1983) 135.
[17] K. Nozaki and N. Bekki, Phys. Rev. Let. 51 (1983) 2171.
[18] L.R. Keefe, Dynamics of perturbed wavetrain solutions to the Ginzburg-Landau equation, Ph.D. Thesis (1984), to be published in Stud. App. Math.
[19] Y. Kuramoto and S. Koga, Phys. Lett. A 92 (1982) 1.
[20] K. Stewartson and J.T. Stuart, J. Fluid Mech. 48 (1971) 529.
[21] P.J. Blennerhassett, Phil. Trans. R. Soc. A 298 (1980) 43.
[22] A.C. Newell and J.A. Whitehead, J. Fluid Mech. 38 (1969) 279.
[23] S. Kogelman and R.C. Di Prima, Phys. Fluids 13 (1970) 1.
[24] T. Watanabe, J. Phys. Soc. Japan 27 (1969) 1341.
[25] A. Wolf, J.B. Swift, H.L. Swinney and J.A. Vastano, Physica D 16 (1985) 285;
J. Wright, Phys. Rev. A 29 (1984) 2923;
M. Sano and Y. Sawada, Phys. Rev. Lett. 55 (1985) 1082.
[26] J. Crutchfield, Noisy chaos, Ph.D. Thesis (1983).
[27] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480; Ph.D. Thesis (1983), to be published (World Scientific, Singapore).
[28] K. Kaneko, Prog. Theor. Phys. 74 (1985) 1033.
[29] J. Crutchfield and K. Kaneko, Statistical mechanics of lattice dynamical systems, in preparation.
[30] R. Kapral, Phys. Rev. A 31 (1985) 3868.
[31] Y. Aizawa, Prog. Theor. Phys. 72 (1984) 662.
[32] T. Yamada and H. Fujisaka, Prog. Theor. Phys. 73 (1985).
[33] J. Keeler and D. Farmer, in preparation.
[34] K. Kaneko, Phys. Lett. A 111 (1985) 321.