Supertransients, spatiotemporal intermittency and stability of fully developed spatiotemporal chaos

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Structural stability of fully developed spatiotemporal chaos (FDSTC) is confirmed. The stability is sustained by the destruction of all windows through spatiotemporal intermittency and supertransients. Transition to intermittency is found to occur successively, via type-I and then type-II transients. In type-I transients, the transient time increases algebraically with system size, while it increases exponentially in type-II. Decay of positive Lyapunov exponents is gradual in the former and abrupt in the latter. In FDSTC, spatial and temporal correlations are found to decay exponentially, as measured by mutual information. The existence of a finite correlation length assures the density of thermodynamic quantifiers.

Spatiotemporal chaos is a complex dynamical phenomenon with many degrees of freedom, emerging in spatially extended systems. It appears in a broad area of natural phenomena. Qualitative, quantitative, and theoretical understanding of spatiotemporal chaos remains one of the most important problems in nonlinear dynamics.

As a simple model for spatiotemporal chaos, coupled map lattices (CML) have been proposed [1-3].

A CML is a dynamical system with a discrete time, discrete space, and continuous state [1-14]. Although there are various types of CML, we restrict ourselves here to the following diffusive coupling case here:

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i)) + \tfrac{1}{2}\epsilon[f(x_n(i+1))+f(x_n(i-1))]\,, \qquad (1)$$

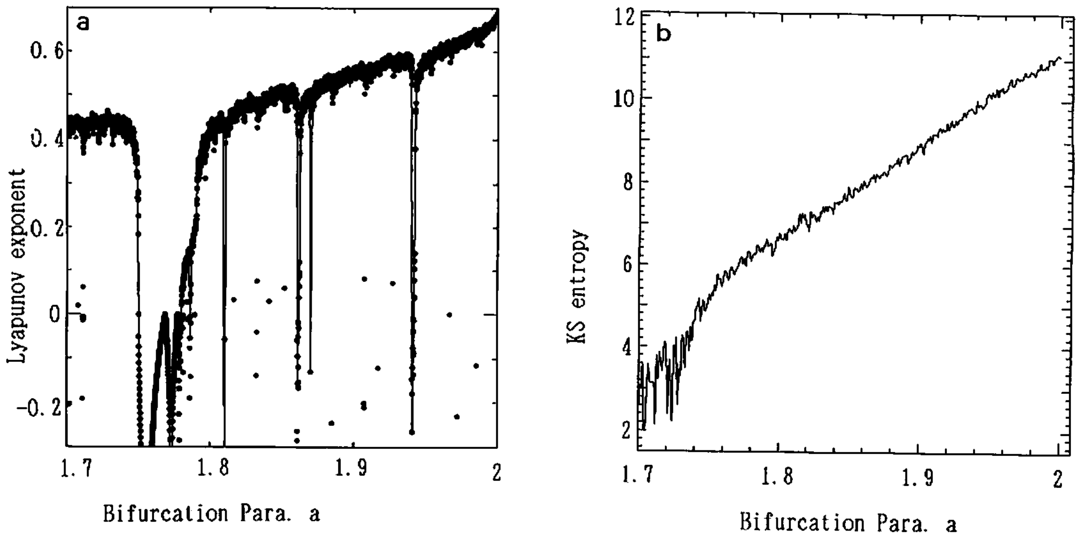
where n is a discrete time step and i is a lattice point ($i=1, 2, ..., N=$ system size) with a periodic boundary condition. Here the mapping function $f(x)$ is chosen to be the logistic map $f(x)=1-ax^2$ (coupled logistic lattice). Results to be presented here are applied to other maps and other couplings.

Separation of procedures is essential in the construction of CML [1,4,11]. In the above model, local nonlinear transformation and diffusion processes are separated: the dynamics consists of $x_n(i)\to x'(i)=f(x_n(i))$ and then applying the discrete Laplacian operator $x_{n+1}(i)=(1-\epsilon)x'(i)+\frac{1}{2}\epsilon[x'(i+1)+x'(i-1)]$.

CML has originally been introduced to model turbulent behavior as a synthesis of Landau's picture on turbulence [15] and Rössler's hyperchaos [16]. Landau has regarded turbulence as a direct product of periodic states (quasiperiodic state with many incommensurate frequencies). This direct product state, however, is not stable. It is easily locked to a lower-dimensional torus [2], or attracted to a nearby strange attractor [17,2].

Not only a high-dimensional quasiperiodic state but also a low-dimensional chaotic state is structurally unstable. In the logistic map, for example, chaos cannot exist in an open set in the parameter space [18]. Periodic windows are dense, although the chaotic state also has positive measure in the parameter space. Quantifiers such as the Lyapunov exponent have infinitely many drops, if plotted as a function of the bifurcation parameter (see fig. 1a). The chaotic state is structurally unstable, even if it observable.

On the other hand, a direct product of chaos ((hyper)$^\infty$chaos [13]) may be structurally stable and may provide a metaphorical model for turbulence. In the



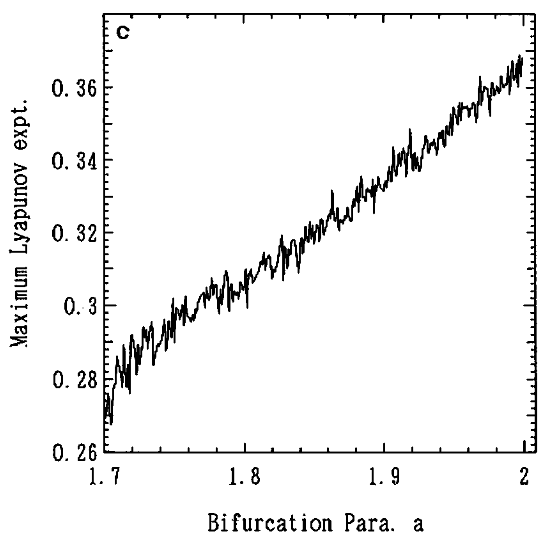


Fig. 1. (a) Parameter dependence of the Lyapunov exponent of the single logistic map. The solid line is drawn for the Lyapunov exponent with the change of bifurcation parameter a by 10^{-3}, while the circles indicate the exponent obtained by the change by 10^{-4}. Note that the increase of resolution leads to the increase of drops (windows). (b) Kolmogorov–Sinai entropy and (c) maximal Lyapunov exponent of our logistic lattice (1), as a function of a, calculated from the Lyapunov spectra. The spectra are obtained from 3000 steps after discarding 2000 transients, starting from a random initial condition. $\epsilon=0.2$, and $N=64$.

present Letter we confirm the stability of (hyper)$^{\infty}$chaos.

As has been investigated in detail [1,4], our CML exhibits the successive phase changes as: (i) period doubling of kinks, (ii) frozen random pattern with localized chaos, (iii) pattern selection, (iv) spatiotemporal intermittency transition, and (v) fully developed spatiotemporal chaos (FDSTC).

We address the following questions here: Is the above picture of structural stability valid in FDSTC? If so, what is the mechanism to sustain the stability and to mask the window structures in low-dimensional chaos? The answers we present here are: FDSTC is structurally stable. All windows are destroyed for almost all initial conditions. Mechanisms of this destruction are spatiotemporal intermittency [1,4,7] and supertransients [8].

Numerical evidence to support the structural stability is as follows.

In FDSTC, all quantifiers change smoothly with parameters for almost all initial conditions. In figs. 1b, 1c, we have plotted the Kolmogorov–Sinai (KS) entropy and maximal Lyapunov exponent as a function of the bifurcation parameter a. Figs. 1b, 1c are in strong contrast with fig. 1a for Lyapunov exponents of single logistic maps. In the logistic map, periodic windows are dense. Both chaos and periodic windows have positive measure in the parameter space. Chaos cannot exist in an open set in the parameter space.

Why do windows in the logistic map disappear in coupled systems? If $x=x_j^*$ ($j=1, 2, ..., p$) is a stable cycle of period p for the logistic map in a window, the spatially homogeneous and temporally periodic state $x(i)=x_j^*$ is linearly stable, as can be easily checked. This window should be observed for initial conditions in the vicinity of the homogeneous state. For almost all initial conditions, however, windows are not observed. This fact suggests that the basin volume of such a homogeneous attractor may be small and/or that the transient time before our system is attracted into the homogeneous attractor may be extremely long.

Since any window in the logistic map is created by the essentially identical mechanism, we consider the simplest case here: a period-three window in the logistic map. The period-three cycle is stable for $1.75<a<1.76...$ for the logistic map. In this parameter regime $x'=f^3(x)$ has three stable fixed points x_1^*, x_2^*, x_3^*, and three unstable fixed points x_1', x_2' and x_3'. The former set gives the stable period-three cycle while the latter gives the unstable period-three cycle of the logistic map (see fig. 2).

In our logistic lattice the homogeneous period-three state ($x(i)=x_j^*$) is stable for this parameter range. However, the basin volume ratio for such homogeneous state is extremely small. We have never encountered the attraction into this homogeneous state, from arbitrarily chosen initial conditions, if the size is large.

Let us define the interval I_j as $I_j \equiv [x_j^*, x_j']$ (see fig. 2). If all of the $x_0(i)$ belong to a unique interval I_k, our system is attracted into the homogeneous state within few time steps. The basin volume for this condition is $\sum_{j=1}^{3} |x_j'-x_j^*|^N$, which decreases exponentially with the system size.

On the other hand, if our initial condition $x_0(i)$ covers at least two intervals I_i and I_j ($j\neq i$), there exists at least a kink initially which separates two domains with different phases of oscillations. Since the logistic map gives transient chaos in this value range, the lattice point at a kink moves chaotically in time. If the coupling is not extremely small [#1] ($\epsilon>\epsilon_t=6.6\times10^{-4}$) it makes the chaotic motion propagate to neighboring lattice points. Several chaotic defects exist initially.

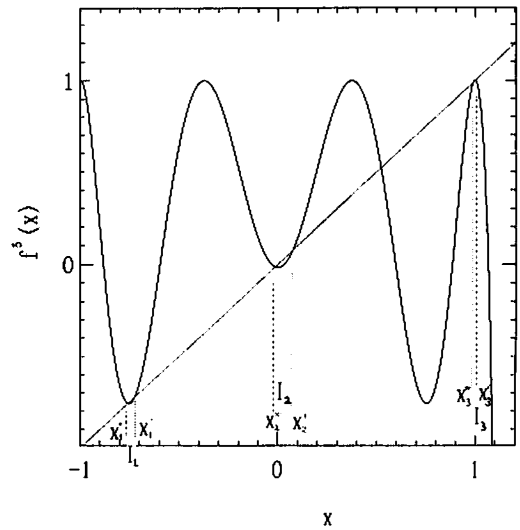


Fig. 2. $f^3(x)$ of the single logistic map. $a=1.762$. If the initial value of x lies in I_j, then x_n is attracted to x_j^*, a fixed point of $f^3(x)$, without any chaotic transients. Chaotic motion exists on a Cantor set between I_j and I_k.

Depending on the coupling strength, we have found two distinct forms of the interaction of defects (fig. 3).

(1) Weak coupling regime ($\epsilon_t<\epsilon<\epsilon_c=10^{-3}$): defects with chaotic motion can never be created. Collision of two defects leads to pair annihilation, merging to a single defect, or preservation of them (figs. 3b, 3c).

(2) Strong coupling regime ($\epsilon>\epsilon_c$): collision of defects can lead to the creation of turbulent bursts besides the above annihilation, merging, or preservation. The creation continues, as long as there are some defects (see fig. 3d). This creation and annihilation of bursts lead to *spatiotemporal intermit-*

[#1] Hereafter the parameter a is fixed at 1.752. The transition couplings ϵ_t and ϵ_c depend on a.

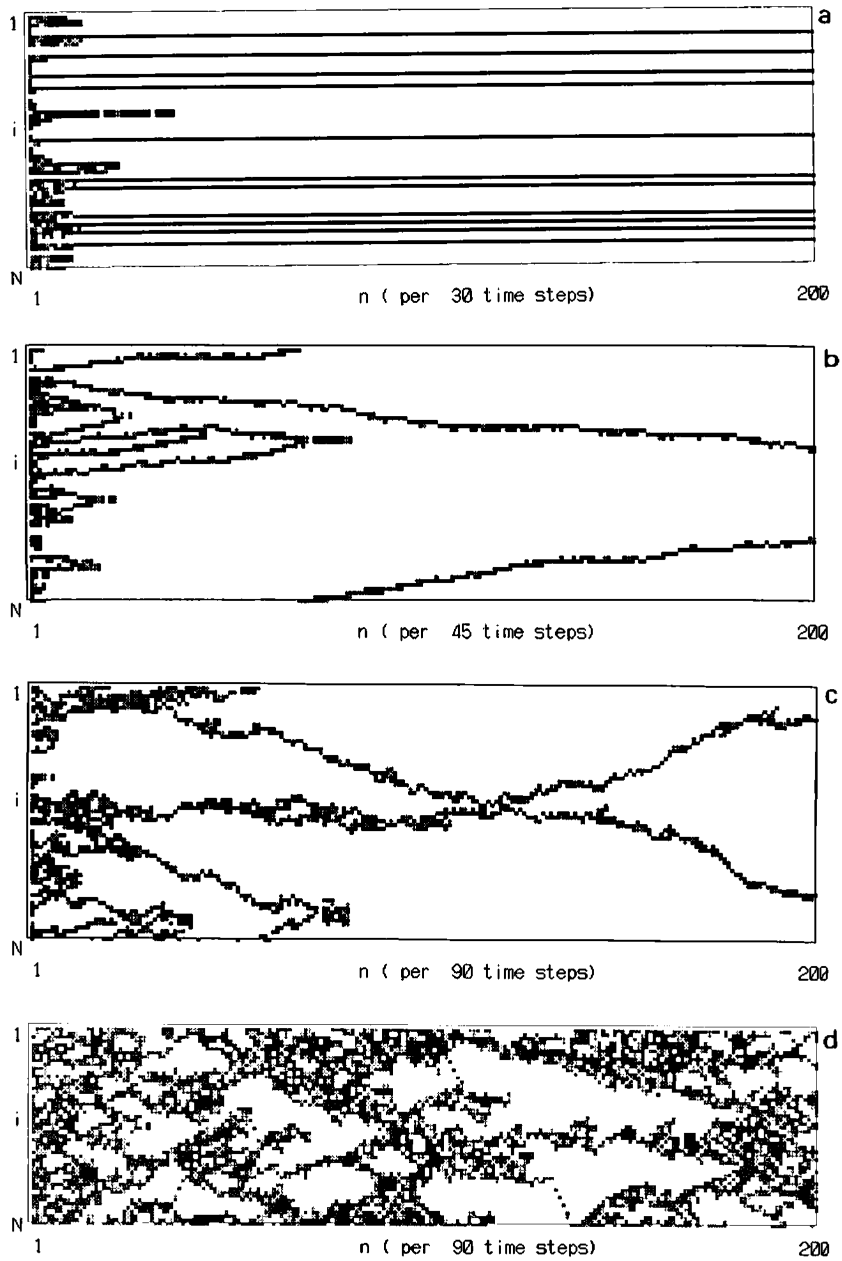


Fig. 3. Spatial derivative plots for the coupled logistic lattice (1), starting with a random initial condition. If $|x_n(i+1)-x_n(i)|$ is larger than 0.3, the corresponding space–time pixel is painted as black, as gray if it is between 0.1 and 0.3, while it is left blank otherwise. (a) $\epsilon=6.6\times10^{-4}$ and $N=64$ (plotted every 30th step), (b) $\epsilon=7.0\times10^{-4}$ and $N=64$ (plotted every 45th step). Around the time step 1.5×10^4, the two defects pair-annihilate and our system is attracted into the homogeneous period-three cycle. (c) $\epsilon=9\times10^{-4}$ and $N=64$ (plotted every 90th step). Around the time step 4×10^4, the defects disappear and our system is attracted into the homogeneous period-three cycle. (d) $\epsilon=1.05\times10^{-3}$ and $N=50$ (plotted every 90th step).

tency [1], as has extensively been investigated recently [4,7] (see for some experiments on spatiotemporal intermittency ref. [19]).

In the latter case, we have sustained spatiotemporal intermittency in so far as we do not start from an exceptional initial condition such that $x(i) \in$ single I_j for $\forall i$. Since the measure for such initial conditions decreases exponentially with the size, it is concluded that windows in local dynamics are extinguished by spatiotemporal intermittency in a coupled system.

So far we have used the term "basin" to mean the set of initial conditions which are attracted to a given state within *non-astronomical* time steps, less than the order of $\exp(\text{const} \times N)$. If we wait for a very long time, can we expect that our system finally hits the homogeneous state? In other words, does our system exhibit supertransients as discussed in ref. [8]?

In fig. 4, we have plotted the average transient time steps before our system is attracted into a period-three state, as a function of system size N.

For $\epsilon < \epsilon_t$, the transient length does not increase with the system size. Within $o(N)$ time steps our system reaches a direct product state with kinks and stays there (fig. 3a).

For $\epsilon_t < \epsilon < \epsilon_c$, the transient length increases with the system size with the form N^σ. The exponent σ is fitted by 2. The increase form N^2 is expected if our dynamics is replaced by Brownian motion of pair-annihilating defects [4].

For $\epsilon > \epsilon_c$, the transient length increases with the system size as

$$T_N = \text{const} \times \exp(rN) . \tag{2}$$

The coefficient r of the exponential divergence of the transients increases with ϵ as

$$r \propto (\epsilon - \epsilon_c)^\gamma . \tag{3}$$

From our data the exponent γ is estimated to be 0.9 ± 0.3. It is related with the exponent for the correlation length. The exponential divergence of the transient length is understood as follows: If our system has a finite correlation length ξ, two elements farther distant than ξ move essentially independently. Then our system is divided into N/ξ subunits. As a simple approximation we can assume that each subunit takes either the laminar (L) or the turbulent (T) state. As long as there exists a turbulent site, it can propagate into neighboring sites. If all of our lattice points happen to hit the laminar state (LLL...LLL), our system remains in the LLL...LLL state. Thus the transient time for the attraction into the periodic state is approximated by the average waiting time for the search for the LLL...LLL state. Using the single-point probability p_L that an element takes the laminar state, and neglecting the correlation in neighboring subunits, the probability that all elements take L is given by $p_L^{N/\xi}$. The average time to find the state LLL...LLL is inversely proportional to the above probability. Thus the transient length T_N obeys

$$T_N \propto p_L^{-N/\xi} \propto \exp(C_t N) . \tag{4}$$

The constant C_t is roughly estimated through the probability that each lattice falls on a laminar state. It is given by

$$p_L = \int_{x \in \Sigma_j I_j} dx \, \rho(x)$$

by using the quasistationary probability measure that a single lattice point takes a value x.

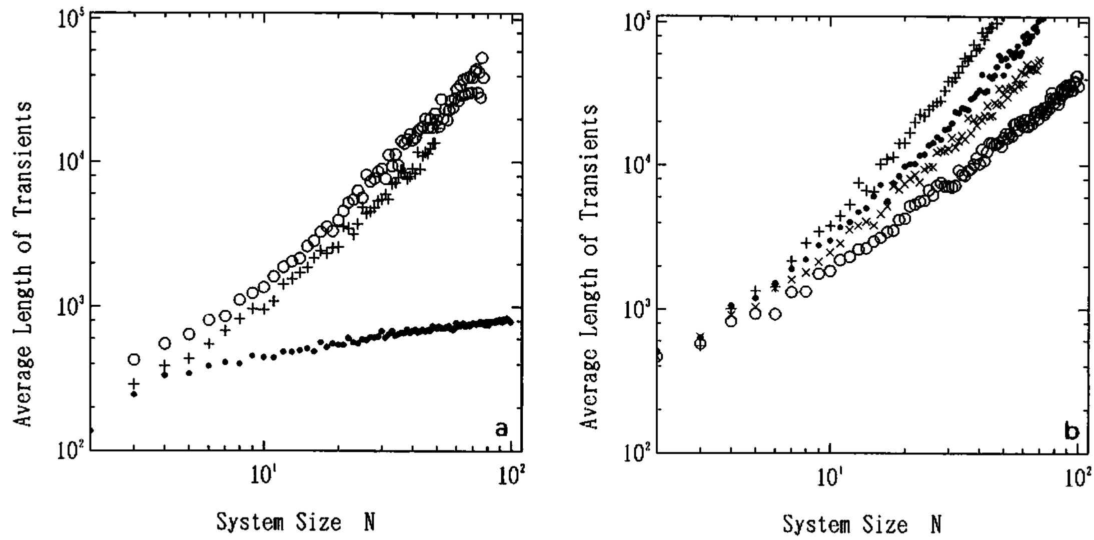
If we follow the commonsense of critical phenomena, the correlation length ξ diverges as $\xi \propto (\epsilon - \epsilon_c)^{-\nu}$. Our critical exponent γ of transients is expected to be equal to the exponent ν [#2].

Near $\epsilon \approx \epsilon_c$, the transient length increases with some power: $T_N = N^\eta$. This power-law dependence is also expected from the power-law decay of the correlation at the transition point.

In fig. 3, the spacetime diagram for a small lattice is shown. At $\epsilon_t < \epsilon < \epsilon_c$, the number of chaotic bursts decreases with time. By the collision, these bursts disappear successively. For $\epsilon > \epsilon_c$ the chaotic state seems to be quasistationary with typical spatiotemporal intermittency [1,7] before our system is finally attracted into the homogeneous period-three state.

To see clearly the temporal change of chaotic strength and degrees, we have plotted the maximal Lyapunov exponent and KS entropy over short time averages, as a function of time. The maximal Lyapunov exponent is calculated by the average over given M time steps, without taking the $M \to \infty$ limit.

[#2] For comparison, $\nu \approx 1.1$ for directed percolation.



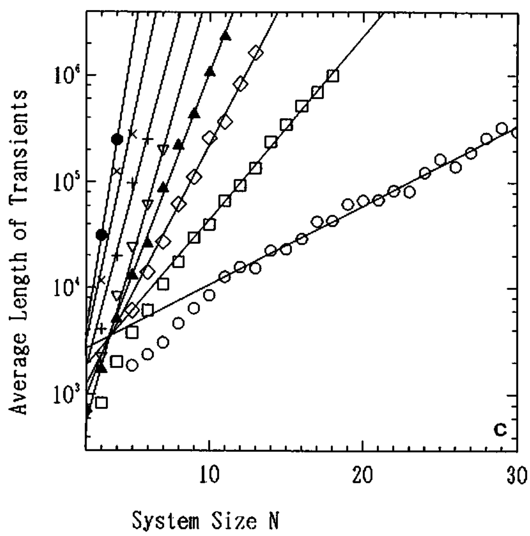


Fig. 4. Average transient length versus system size N. Average time steps before our system is attracted into a period-three cycle are calculated from 100 randomly chosen initial conditions. $a=1.752$. (a) Log-log plot for $\epsilon=0.00065$ (●), 0.0007 (o), and 0.00075 (+). (b) Log-log plot for $\epsilon=0.00088$ (o), 0.00092 (×), 0.00096(●), and 0.001 (+). (c) Semi-log plot for $\epsilon=0.0011$ (o), 0.0012 (□), 0.0013 (◇), 0.0014 (▲), 0.0015 (▽), 0.002 (+), 0.003 (×), and 0.005 (●). The slopes are 0.074 ($\epsilon=0.0011$), 0.17 ($\epsilon=0.0012$), 0.29 ($\epsilon=0.0013$), 0.38 ($\epsilon=0.0014$), 0.47 ($\epsilon=0.0015$), 0.76 ($\epsilon=0.002$), 1.0 ($\epsilon=0.003$), and 1.4 ($\epsilon=0.005$).

The KS entropy is calculated by the sum of positive Lyapunov exponents over the same finite M steps. For $\epsilon<\epsilon_c$, we have observed successive decrease of chaotic degrees as is seen in the change of KS entropy (fig. 5a). The maximal Lyapunov exponent, on the other hand, does not exhibit such successive change.

For $\epsilon>\epsilon_c$, the existence of a quasistationary measure is seen (fig. 5b). The escape from the quasistationary state is abrupt without any gradual decay [20].

The above scaling, the spatial derivative, and Lyapunov exponents lead us to the conclusion that the dynamics in the second regime ($\epsilon_1<\epsilon<\epsilon_c$) belongs to type-I supertransients, while that in the third regime ($\epsilon>\epsilon_c$) belongs to type-II supertransients.

The distinction of type I/II supertransients is summarized as follows (see ref. [8] for details).

In type-I supertransients, (1) the length of the transient time diverges with system size slower than exponentially; and (2) the motion is chaotic but the degree of chaos (e.g., the KS entropy or the number

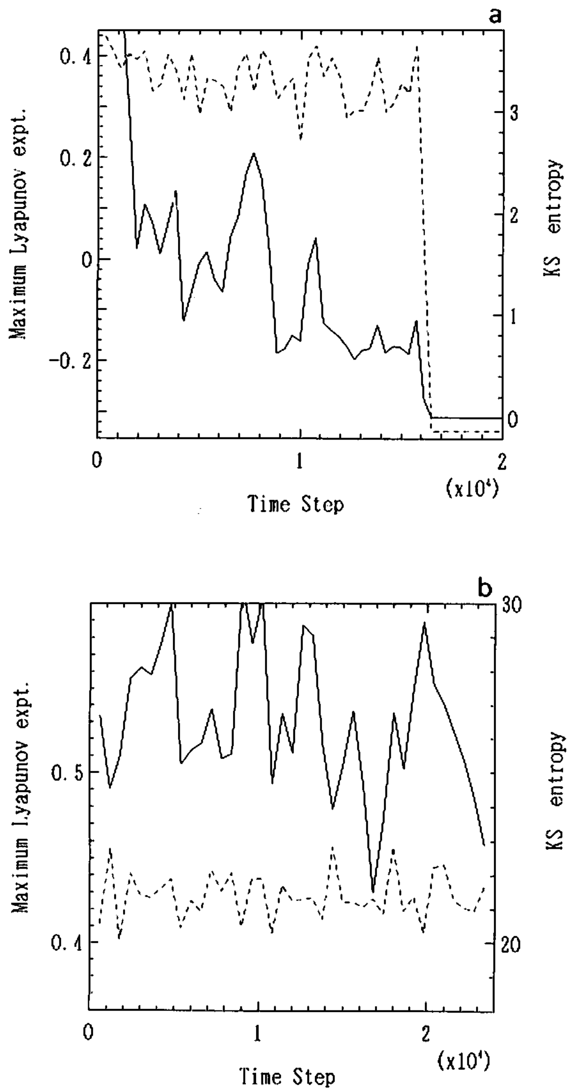


Fig. 5. Maximal Lyapunov exponent (dashed line) and KS entropy (solid line) over short time steps are plotted as a function of time for our logistic lattice (1). Using the N-dimensional Jacobi matrix J of our lattice dynamics, the short-time Lyapunov exponent $\lambda(t)$ is defined by the logarithm of the largest eigenvalue of $\prod_{j=1}^{M} J_{Mt+j}$, $a=1.752$, and $N=30$ and starting with a random initial condition. (a) $\epsilon=0.0009$, $N=64$ and $M=384$. (b) $\epsilon=0.0012$, $N=100$ and $M=600$.

of positive Lyapunov exponents) decreases with time.

In type-II, on the other hand, (1) the length of the transient time diverges with system size exponentially or faster; (2) the motion is chaotic with many positive Lyapunov exponents without any decay; (3) there exists a quasi-stationary measure; and (4) the termination of transients is abrupt.

In the present Letter, the stability of FDSTC is confirmed. Windows of local dynamical systems are destroyed by spatiotemporal intermittency. The intermittency is sustained as supertransients. Successive transition from type-I to type-II supertransients is found with the increase of coupling.

If we couple local dynamical systems with topological chaos and periodic attractors, such a coupled system is expected to show spatiotemporal intermittency. Topological chaos becomes observable in a coupled system as long as the coupling is not extremely small.

In FDSTC, spatial and temporal correlations decay exponentially, as measured by mutual information [20]. The rate of the exponential decay decreases with the nonlinearity, till it goes to zero at the intermittency transition. If the correlation length is given by ξ, two points more distant than the correlation length ξ move independently. Our system is roughly approximated by a direct product of N/ξ independent systems. We can approximately replace our dynamics by local chaos in a subspace and the heat bath from boundaries. Then the number of positive Lyapunov exponents, Kolmogorov–Sinai entropy, and the dimension of attractors are expected to be proportional to the system size N [3,4,10]. This existence of extensive quantifiers assures the well-defined existence of thermodynamic densities. It will be promising and important to construct the thermodynamics for fully developed spatiotemporal chaos [9,11,20].

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References

[1] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480; 74 (1985) 1033; in: Dynamical problems in soliton systems, ed. S. Takeno (Springer, Berlin, 1985) pp. 272–277.

[2] K. Kaneko, Ph.D. Thesis, Collapse of tori and genesis of chaos in dissipative systems (1983) [enlarged Ed. (World Scientific, Singapore, 1986)].

[3] K. Kaneko, Physica D 23 (1986) 436.

[4] K. Kaneko, Physica D 34 (1989) 1; 37 (1989) 60; Europhys. Lett. 6 (1988) 193; Phys. Lett. A 125 (1987) 25.

[5] J.P. Crutchfield and K. Kaneko, in: Directions in chaos (World Scientific, Singapore, 1987) pp. 272–353.

[6] R.J. Deissler, Phys. Lett. A 120 (1984) 334;
I. Waller and R. Kapral, Phys. Rev. A 30 (1984) 2047;
K. Kaneko, Phys. Lett. A 111 (1985) 321;
R.J. Deissler and K. Kaneko, Phys. Lett. A 119 (1987) 397;
Y. Oono and S. Puri, Phys. Rev. Lett. 58 (1986) 836;
M.H. Jensen, Phys. Rev. Lett. 62 (1989) 1361;
T. Bohr and O.B. Christensen, Phys. Rev. Lett. 63 (1989) 2161.

[7] J.D. Keeler and J.D. Farmer, Physica D 23 (1986) 413;
H. Chate and P. Manneville, Europhys. Lett. 6 (1988) 59; Physica D 32 (1988) 409.

[8] J.P. Crutchfield and K. Kaneko, Phys. Rev. Lett. 60 (1988) 2715.

[9] L.A. Bunimovich and Ya.G. Sinai, Nonlinearity 1 (1989) 491.

[10] G. Mayer-Kress and K. Kaneko, J. Stat. Phys. 54 (1989) 1489.

[11] K. Kaneko, Phys. Lett. A 139 (1989) 47;
J.M. Houlrik, I. Webman and M.H. Jensen, Phys. Rev. A 41 (1990) 4210.

[12] K. Kaneko, in: Formation, dynamics, and statistics of patterns, eds. K. Kawasaki, A. Onuki and M. Suzuki (World Scientific, Singapore, 1990) pp. 1–52.

[13] K. Kaneko, Climbing up dynamical hierarchy, in: Chaotic hierarchy, eds. G. Baier and M. Klien (World Scientific, Singapore, 1990).

[14] K. Kaneko, Phys. Rev. Lett. 63 (1989) 219; Physica D 41 (1990) 137.

[15] L.D. Landau and E.M. Lifshitz, Fluid mechanics (Pergamon, Oxford, 1959) ch. 3.

[16] O.E. Rössler, Phys. Lett. A 71 (1979) 155; in: Lecture notes in applied mathematics, Vol. 17 (Springer, Berlin, 1979) pp. 141–156; Z. Naturforsch. 38a (1983) 788.

[17] D. Ruelle and F. Takens, Commun. Math. Phys. 20 (1971) 167; 23 (1971) 343.

[18] P. Collet and J.P. Eckmann, Iterated maps on the intervals as dynamical systems (Birkhäuser, Basel, 1980);
J.D. Farmer, Phys. Rev. Lett. 55 (1985) 351.

[19] S. Ciliberto and P. Bigazzi, Phys. Rev. Lett. 60 (1988) 286;
S. Nasuno, M. Sano and Y. Sawada, in: Cooperative dynamics in complex systems, ed. H. Takayama (Springer, Berlin, 1989);
F. Daviaud, M. Dubois and P. Berge, Europhys. Lett. 9 (1989) 441;
J. Gollub and R. Ramshankar, in: New perspectives in turbulence, eds. S. Orszag and L. Sirovich (Springer, Berlin), to be published.

[20] K. Kaneko, Prog. Theor. Phys. Suppl. 99 (1990) 263.