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# Propagation of disturbance, co-moving Lyapunov exponent and path summation

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Estimates for the co-moving Lyapunov exponents are presented for a class of coupled map lattices with the use of path summation of disturbances. The propagation speed of the applied disturbance is estimated from the exponents, which agrees with numerical results for coupled piecewise-linear and tent maps.

Chaotic dynamics amplifies a tiny disturbance to a macroscopic level as is quantified by the Lyapunov spectrum. In spatiotemporal chaos, a small disturbance, applied locally on a spatial region, can be amplified (in a bit space) and transmitted (in real space). The amplification in a Galilean frame is quantified by the co-moving Lyapunov exponent $L(v)$ [1,2]. The exponent is also useful to estimate the propagation speed of the disturbance and the mutual information flow.

Recall the space–time difference patterns of refs. [2,3], where we measure the propagation of the disturbance by applying a tiny perturbation at a lattice point. If an applied disturbance spreads without bound, the spread gives the propagation speed $v_*$. The zero-crossing point of the co-moving Lyapunov exponent ($v_*$ such that $L(v_*)=0$) corresponds to the above propagation limits of the disturbance bands. Thus, the co-moving Lyapunov exponent is useful to determine the velocity band where the disturbance is allowed to be transmitted.

In general it is rather difficult to estimate the co-moving Lyapunov exponents from other dynamical quantifiers. In the present Letter we introduce a path summation method through which the co-moving Lyapunov exponents and propagation speed of the disturbance are estimated. As an explicit illustration we choose a coupled map lattice (CML) model [1-8] given by

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\tfrac{1}{2}\epsilon[f(x_n(i))+f(x_n(i-1))] , \qquad (1)$$

although the method to be presented is straightforwardly extended to CMLs of other coupling types, and possibly to coupled differential equations and to partial differential equation systems. We explain our calculation using piecewise-linear maps ($f(x)=\beta x \pmod 1$) and tent maps ($f(x)=\beta(0.5-|0.5-x|)$), and also discuss an extension to a general CML.

Let us consider the evolution of a small disturbance applied on a lattice site 0. The evolution is given by the difference between the dynamics $x_n(i)+\delta x_n(i)$ and $x_n(i)$, each of which is governed by the CML (1). The deviation at time $n$ and site $i$ is given by the following sum over the paths (see fig. 1),

$$\delta x_n(i)=\sum_{i1,i2,\ldots,in}\prod_k\{f'(x_k(i_k))[(1-\epsilon)\delta_{ik,ik+1}+\tfrac{1}{2}\epsilon(\delta_{ik,ik+1+1}+\delta_{ik,ik+1-1})] . \qquad (2)$$

For this summation, we introduce the following generating function,

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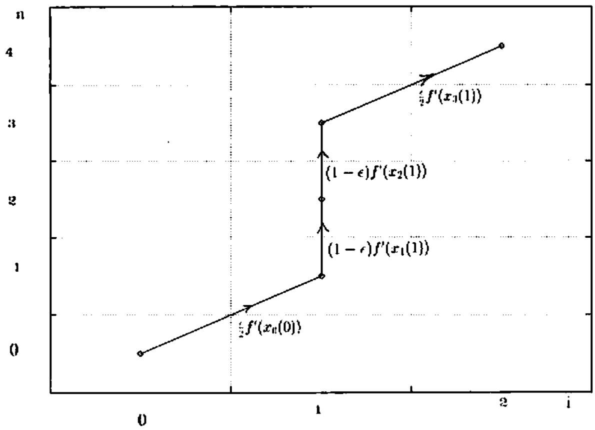


Fig. 1. Schematic diagram of the path summation.

$$\prod_{m=1}^{m=n} \{f'(x_m(i_m*))[(1-\epsilon)+\tfrac{1}{2}\epsilon(zR+z^{-1}L)]\}f'(x_0(i_0)) , \tag{3}$$

where $R$ and $L$ are shift operators to move the lattice site to the right $(i\to i+1)$ or to the left $(i\to i-1)$, and $i_m*$ is the current lattice point obtained by these operators, starting from the site 0.

The propagated amplitude of a disturbance at the site $k$ is given by the coefficient of $z^k$, since the term corresponds to the shift of $k$ lattice points during $n$ time steps. Setting $k=vn$, we can obtain the disturbance propagation with speed $v$. Since there is no term with $|k|>n$ in the expansion of eq. (3), the disturbance does not propagate faster than the speed of the "light cone", as is expected. To get the amplitude at the site $k=vn$ $(<n)$, we need the expansion of $[(1-\epsilon)+\frac{1}{2}\epsilon(z+z^{-1})]^n$ as

$$\sum_{0\leqslant n_1,n_2,n_1+n_2\leqslant n} \frac{n!}{n_0!n_1!n_2!} (1-\epsilon)^{n-n_1-n_2}(\tfrac{1}{2}\epsilon)^{n_1+n_2} , \tag{4}$$

for $n_1=n_2+vn$. For the calculation of the co-moving Lyapunov exponent one only needs the asymptotic behavior for large $n$ in the expression.

For the calculation we have to multiply the corresponding $f'(x_m(i_m))$ term. As the simplest case we first analyze the piecewise-linear map with a constant expansion rate $\beta$, in which case the term is just $\beta^n$. Using the Stirling formula, $x!\approx\exp[x\log(x)]$, the factor is given by

$$\sum_{0\leqslant n_2\leqslant(1-v)n/2} \frac{n!}{[(1-v)n-2n_2]!(n_2+vn)!n_2!} (1-\epsilon)^{(1-v)n-2n_2}(\tfrac{1}{2}\epsilon)^{2n_2+vn}\beta^n \tag{5}$$

$$\approx\beta^n \sum_{0\leqslant K\leqslant(1-v)/2} (1-\epsilon)^{(1-v-2K)n}(\tfrac{1}{2}\epsilon)^{(v+2K)n}$$

$$\times\exp\{-(K+v)n\log[(K+v)n]-Kn\log(Kn)+n\log(n)-(1-v-2K)n\log[(1-v-2K)n]\} \tag{6}$$

$$\approx \sum_{0\leqslant K\leqslant(1-v)/2} \{(1-v-2K)^{-(1-v-2K)}K^{-K}(K+v)^{-(K+v)}(1-\epsilon)^{1-v-2K}(\tfrac{1}{2}\epsilon)^{v+2K}\beta\}^n , \tag{7}$$

with the use of the transformation $n_2=Kn$. As is easily seen in the derivation, the term $\beta(1-\epsilon)^{1-v-2K}(\frac{1}{2}\epsilon)^{v+2K}$ gives the amplification by the chaotic dynamics, while the term

$$Z_v(K)=(1-v-2K)^{-(1-v-2K)}K^{-K}(K+v)^{-(K+v)} \tag{8}$$

gives the number of such paths with $K$ turns. If an analogy with the thermodynamics is admitted, the first term $(\beta(1-\epsilon)^{1-v-2K}(\frac{1}{2}\epsilon)^{v+2K})$ corresponds to the energy, and the second term $(Z_v(K))$ corresponds to the entropy associated with the path variety. The dominant behavior of the disturbance amplification is given by the maximum of

$$F(K) \equiv Z_v(K)\beta(1-\epsilon)^{1-v-2K}(\tfrac{1}{2}\epsilon)^{v+2K}$$

under the condition $0 \leqslant K \leqslant (1-v)/2$. By solving $F'(K)=0$, the maximum is given by

$$K_m = \frac{(1-v)f+v-\sqrt{v^2(1-f)+f}}{2(f-1)},$$

with $f=[\epsilon/(1-\epsilon)]^2$.

The co-moving Lyapunov exponent is given by the logarithm of the above amplification ratio, that is

$$L(v) = \log(\beta) + (1-v-2K_m)\log(1-\epsilon) + (v+2K_m)\log(\tfrac{1}{2}\epsilon) + \log[Z_v(K_m)] \tag{9}$$

$$= \log(\beta) + (1-v-2K_m)\log(1-\epsilon) + (v+2K_m)\log(\tfrac{1}{2}\epsilon) - (1-v-2K_m)\log(1-v-2K_m)$$

$$- K_m\log(K_m) - (K_m+v)\log(K_m+v)\ . \tag{10}$$

This exponent $L(v)$, of course, takes its maximum at $v=0$. For $v=0$, $K_m=\frac{1}{2}\epsilon$, and $L(0)=\log(\beta)$ which recovers the calculation of the (conventional) Lyapunov exponent [2,9]. The obtained co-moving Lyapunov exponent from eq. (10) is plotted in fig. 2, which agrees with the numerical results completely. It may also be interesting to see the asymptotic behavior of $\epsilon \to 0$. For small $\epsilon$, $K_m \propto \epsilon^2$, and the co-moving Lyapunov exponent is given by

$$L(v)_{\epsilon\approx 0} = \log(\beta) + (1-v)\log\left(\frac{1-\epsilon}{1-v}\right) + v\log(\tfrac{1}{2}\epsilon/v)\ . \tag{11}$$

If $L(v)$ is positive (i.e., $|\beta(1-\epsilon)^{1-v-2K_m}(\frac{1}{2}\epsilon)^{v+2K_m}Z_v(K_m)| > 1$) the disturbance propagates with the velocity $v$ with amplification. Thus the propagation speed is given by the condition

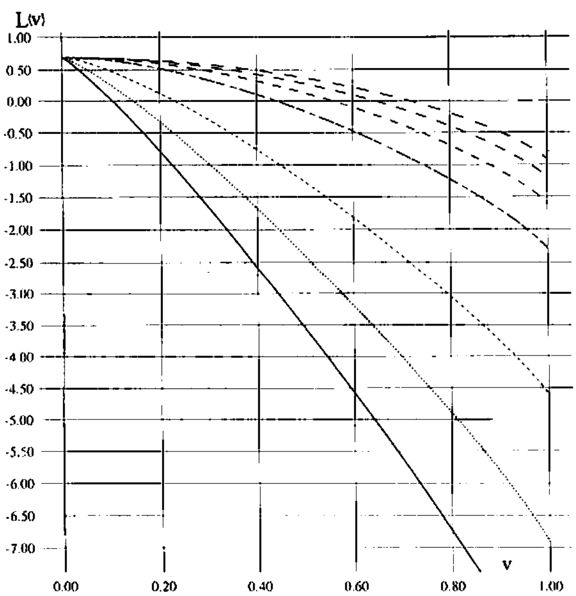


Fig. 2. Co-moving Lyapunov exponent of the coupled piecewise-linear map lattice, obtained from eq. (10), with a complete agreement of the direct numerical computation of the exponent. $\beta=2.0$. The coupling $\epsilon$ is $10^{-4}$, $10^{-3}$, $10^{-2}$, 0.1, 0.2, 0.3, 0.4 from bottom to top.

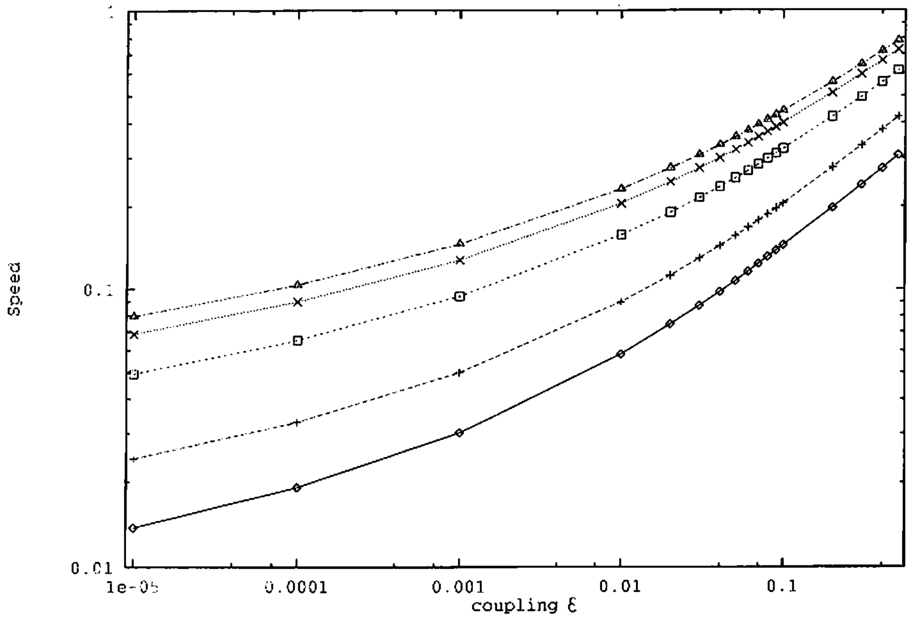


Fig. 3. Propagation speed of the coupled piecewise-linear map lattice as a function of ε obtained from condition (12). $\beta=1.1, 1.3, 1.5, 1.8, 2.0$ from bottom to top.

$$|\beta(1-\epsilon)^{1-v-2K_m}(\tfrac{1}{2}\epsilon)^{v+2K_m}Z_v(K_m)|=1\,. \tag{12}$$

This velocity is shown in fig. 3. The propagation speed is directly obtained numerically from the difference pattern: Perturb one lattice site $J$ to $x_{n0}(J)+\delta$ at time step $n_0$, and perform two independent simulations starting from $x_{n0}(i)$ and $x'_{n0}(i)=x_{n0}(i)+\delta\delta_{i,J}$, and count the number of lattice sites with $|x'_n(i)-x_n(i)|>\delta$. The increase rate of the number with time is given by $2v_p(n-n_0)$, with the propagation speed $v_p$. This propagation speed agrees completely with form (12).

*Tent map.* The extreme simplification in the piecewise-linear map is that the amplification rate is constant. In the second example, we take the coupled tent map lattice, where the absolute value of the expansion rate is constant, but its sign can be positive or negative. Here the above calculation is valid except for the sign. If we neglect the correlation of the sign in the path, we can assume that the probability that the number of points with a negative slope is even (or odd) in a given path is $\frac{1}{2}$. Thus the sign of the amplification rate $\beta(1-\epsilon)^{1-v-2K}(\frac{1}{2}\epsilon)^{v+2K}$ is positive or negative randomly with equal probability. The sign of such paths (whose number is $Z_v(K)$), is positive or negative with equal probability [#1]. Since the root mean square of the sum of $M$ independent variables $\pm 1$ is $\sqrt{M}$, the effective number of paths is given by $\sqrt{Z_v(K)}$ for paths with $K$ turns. Thus the dominant amplification after the path sum is given by the maximum of

$$\beta(1-\epsilon)^{1-v-2K}(\tfrac{1}{2}\epsilon)^{v+2K}\sqrt{Z_v(K)}$$

for $0\leqslant K\leqslant(1-v)/2$. The co-moving Lyapunov exponent is given by

$$L(v)=\log(\beta)+(1-v-2K'_m)\log(1-\epsilon)+(v+2K'_m)\log(\tfrac{1}{2}\epsilon)$$
$$-\tfrac{1}{2}[(1-v-2K'_m)\log(1-v-2K'_m)+K'_m\log(K'_m)+(K'_m+v)\log(K'_m+v)]\,, \tag{13}$$

[#1] Note that we do not need the probability of a positive slope to be $\frac{1}{2}$. We assume that the probability of the multiplication sign of the slopes is equally positive or negative, which holds even if the probabilities of the signs of each step are not equal.

where $K'_m$ has the same form as $K_m$,

$$K'_m = \frac{(1-v)f'+v-\sqrt{v^2(1-f')+f'}}{2(f'-1)},$$

with the use of $f' = \frac{1}{4}[\epsilon/(1-\epsilon)]^4$, instead of the $f$ term for the previous case. The obtained co-moving Lyapunov exponent $L(v)$ is plotted in fig. 4. The agreement with the numerical results is good if $\epsilon$ is not too large (or $a$ is not too small) as will be seen in the following data for the propagation speed.

The propagation speed of the disturbance is again given by the condition $L(v)>0$. The obtained velocity from eq. (13) is plotted in fig. 5, with the estimates by the direct simulations of the difference pattern. The agreement is rather good for large $\beta$ if $\epsilon$ is not large. The discrepancy is due to the correlation in the space-time pattern. Indeed, we have some spatial and temporal pattern structures, which lead to a correlation of the signs for the amplification ratio. The above estimate of the effective number of paths by $\sqrt{Z(K)}$ is based on the assumption of complete decorrelation of signs, and can be lower. The numerically obtained speeds lie between the estimates of (13) and (12), where the sign is always positive (i.e. with complete correlation).

*General case.* In general spatiotemporal chaos, (the absolute value of) the local expansion rate of the orbits is not constant. The amplification rate for a path $i_1, i_2, ..., i_n$ with $K$ turns is given by

$$f'(x_1(i_1))f'(x_1(i_2))f'(x_1(i_3))...f'(x_1(i_n))(1-\epsilon)^{1-v-2K}(\tfrac{1}{2}\epsilon)^{v+2K}.$$

The number of paths with $K$ turns is again given by $Z_v(K)$, but the product of $f'(x_n(i))$ therein fluctuates with the paths. Generally it is rather difficult to estimate this product. A rough approximation is obtained by replacing the amplification ratio $\log(\beta)$ in (13) by the Lyapunov exponent of a single map ($\langle \log|f'(x_n)| \rangle$, with $\langle\ \rangle$ the temporal average). The agreement here is not good; since the order of averaging between the path summation and Lyapunov exponent is not identical. The estimate by $\log[\langle |f'(X)| \rangle]$ gives a slightly better approximation, although the agreement is not yet significant, possibly due to the temporal correlation of the $f'(x)$ term.

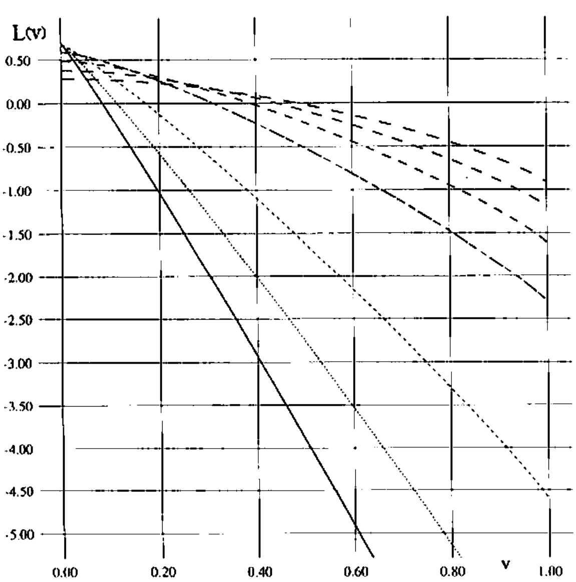


Fig. 4. Co-moving Lyapunov exponent of the coupled tent map lattice, obtained from eq. (13). $\beta=2.0$. The coupling $\epsilon$ is $10^{-4}$, $10^{-3}$, $10^{-2}$, 0.1, 0.2, 0.3, 0.4 from bottom to top. Up to $\epsilon=0.1$ the agreement with the exponent obtained from a direct numerical computation is very good.

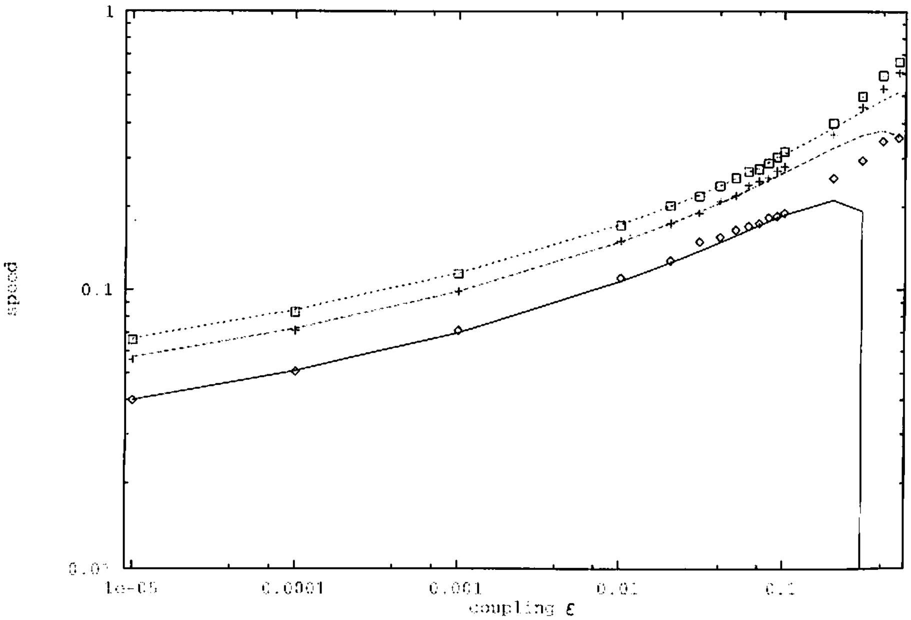


Fig. 5. Propagation speed of the coupled tent map lattice, as a function of $\epsilon$. $\beta$= 1.5, 1.8, 2.0 from bottom to top. The lines are given from expression (13), while (◇), (+), (□) denote the speed obtained by the direct simulation of the CML.

To summarize, we have presented an estimate of the co-moving Lyapunov exponent and the propagation speed of the disturbance, with the use of the path summation method. The estimate agrees completely for coupled piecewise-linear map lattices with a constant slope, and gives a good approximation for coupled tent map lattices. The present method provides a zeroth order approximation, for a general CML case, although further studies will be required in future.

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*Note added.* A similar path summation method is independently applied to the conventional Lyapunov exponent [10].

## References

[1] R.J. Deissler and K. Kaneko, Phys. Lett. A 119 (1987) 397.
[2] K. Kaneko, Physica D 23 (1986) 436.
[3] J.P. Crutchfield and K. Kaneko, Phenomenology of spatiotemporal chaos, in: Directions in chaos (World Scientific, Singapore, 1987).
[4] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480; 74 (1985) 1033.
[5] K. Kaneko, Ph.D. Thesis, Collapse of tori and genesis of chaos in dissipative systems (1983) [enlarged version published by World Scientific, Singapore, 1986].
[6] K. Kaneko, Physica D 34 (1989) 1; 36 (1989) 60.
[7] K. Kaneko, Simulating physics with coupled map lattices – pattern dynamics, information flow, and thermodynamics of spatiotemporal chaos, in: Formation, dynamics and statistics of patterns, ed. K. Kawasaki, A. Onuki and M. Suzuki (World Scientific, Singapore, 1990), and references therein.

[8] I. Waller and R. Kapral, Phys. Rev. A 30 (1984) 2047;
R. Kapral, Phys. Rev. A 31 (1985) 3868.
[9] F. Kaspar and H.G. Schuster, Phys. Lett. A 113 (1986) 451.
[10] R. Livi, A. Politi and S. Ruffo, Scaling law for the maximal Lyapunov exponent, to appear in J. Phys. A;
A. Politi and A. Torcini, Periodic orbits in coupled Hénon maps: Lyapunov and multifractal analysis, to appear in Chaos.