ARNOLD DIFFUSION, ERGODICITY AND INTERMITTENCY IN A COUPLED STANDARD MAPPING

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Numerical studies are performed on a model for high dimensional hamiltonian systems to investigate the Arnold diffusion. We present estimates of the diffusion rate as a function of coupling, the structure of a stochastic layer, calculations of Lyapunov exponents, and some characteristic aspects of a stochastic orbit.

Recently significant progress has been realized in understanding stochasticity in low dimensional hamiltonian systems with the use of two-dimensional area preserving mappings [1–6]. But what of systems with more than two degrees of freedom [7–12]? The purpose of this and subsequent studies is to examine the means of transport in such systems; here we focus on the so called Arnold diffusion. Because N-dimensional invariant (KAM) tori cannot section and hence prevent stochastic trajectories from exploring the $(2N-1)$-dimensional energy surface, it is believed that all stochastic regions on the energy surface are connected. However, there have been few direct observations of Arnold diffusion [1,6,7], the means of transport from one stochastic region to another, and some have disclaimed the possibility of such diffusion in some near integrable systems [12].

One problem is that for near integrable systems, the Arnold diffusion can be observed only within very large time scales. Here we use a discrete mapping to in part avoid this difficulty. Long computation times are still necessary to directly observe the Arnold diffusion, e.g. some simulations require several hours of CRAY time.

A typical method in the study of high dimensional systems is the use of coupled elementary systems such as coupled oscillators [11] [‡1]. We have chosen a coupled standard mapping, which is given by the hamiltonian

$$H=\tfrac{1}{2}\sum_{i=1}^{N}(I^i)^2+\left(\sum_{i=1}^{N}F^i(\theta^i,\ldots,\theta^N)\sum_{n=-\infty}^{\infty}\delta(t-n)\right). \tag{1}$$

In this letter we take $N=2$, and

$$F^i=[k_i/(2\pi)^2]\cos(2\pi\theta^i)+[b/(2\pi)^2]\cos[2\pi(\theta^1+\theta^2)].$$

We define four variables:

$$I=I^1,\quad \theta=\theta^1,\quad J=I^2,\quad \psi=\theta^2.$$

The mapping is given by:

$$\begin{aligned}
I_{n+1}&=I_n+(k_1/2\pi)\sin(2\pi\theta_n)\\
&\quad+(b/2\pi)\sin[2\pi(\theta_n+\psi_n)],\\
\theta_{n+1}&=\theta_n+I_{n+1},\\
J_{n+1}&=J_n+(k_2/2\pi)\sin(2\pi\psi_n)\\
&\quad+(b/2\pi)\sin[2\pi(\theta_n+\psi_n)],\\
\psi_{n+1}&=\psi_n+J_{n+1},
\end{aligned} \tag{2}$$

which was first introduced by Froeschle [11] as a model for the time evolution of elliptical galaxies.

Here we study the case with $k_i < k_c = 0.9716\ldots$

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‡1 For a dissipative system, see ref. [13].

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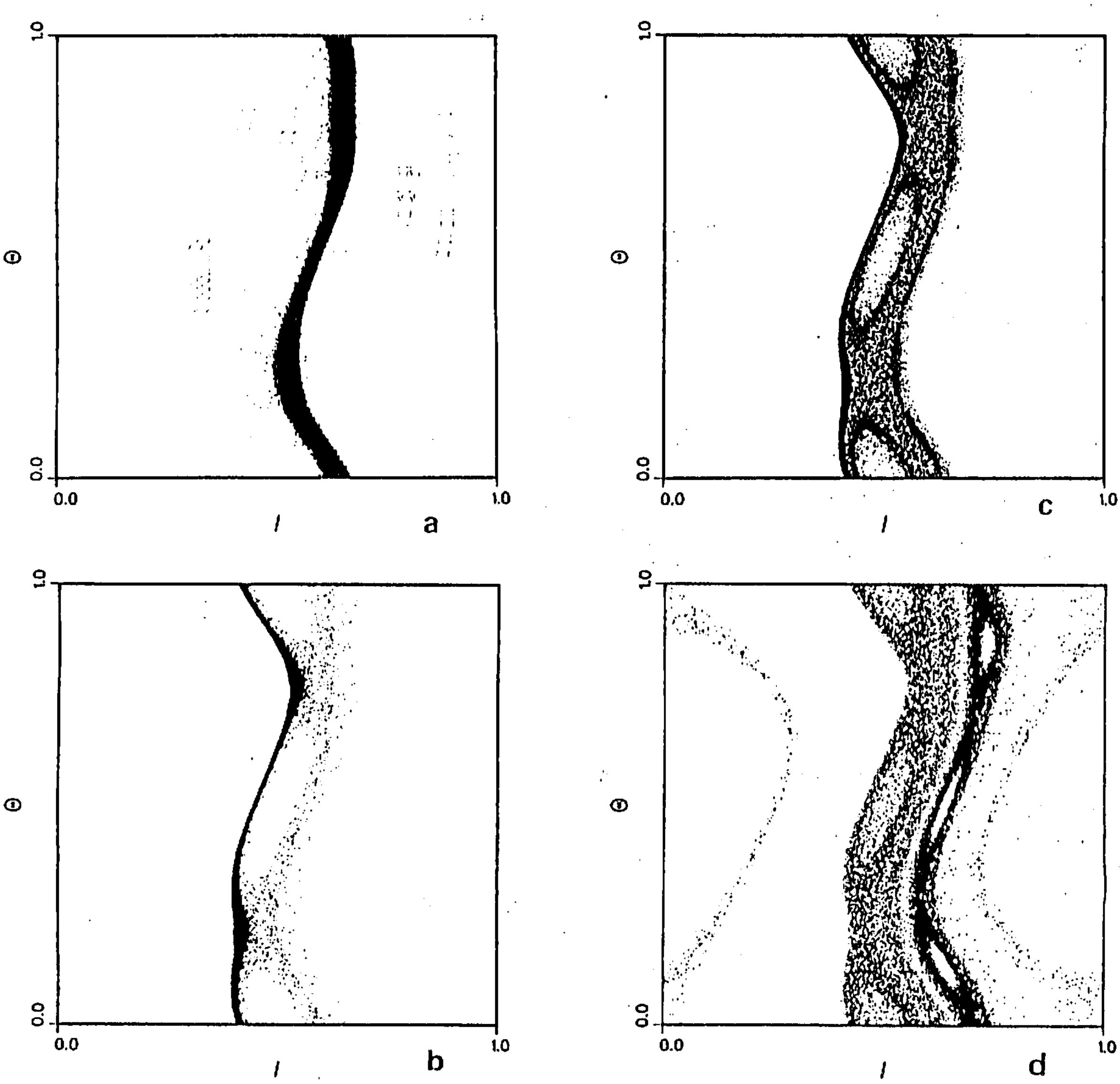
[2,3], where the KAM tori separate the phase space for the two-dimensional standard mapping. Thus, diffusion across KAM tori such as the noble torus is impossible for $b = 0.0$. For $b \neq 0.0$, a stochastic trajectory can reach one KAM bordered region from another by Arnold diffusion. The coupling b is chosen to be small ($b \leqslant 0.08$) so that we may compare the results with the non-coupled system. Figs. 1a, 1b, 1c, and 1d show the points (I_n, θ_n) for $1 \leqslant n \leqslant 2 \times 10^5$ successively. We observe that the orbit started from the stochastic layer close to the $\frac{1}{2}$ resonance remains in the vicinity of a KAM torus (which is close to the direct product of the last KAM tori in the standard mapping) for a long interval (see fig. 1e), and then escapes to the stochastic layer near the 1 resonance.

The time series of (J, ψ) also shows similar behavior. We have also studied some cases with other initial conditions and parameters and have found that the above behavior is rather typical for the coupled standard mapping (2) with $k_I < k_c$, unless the initial value



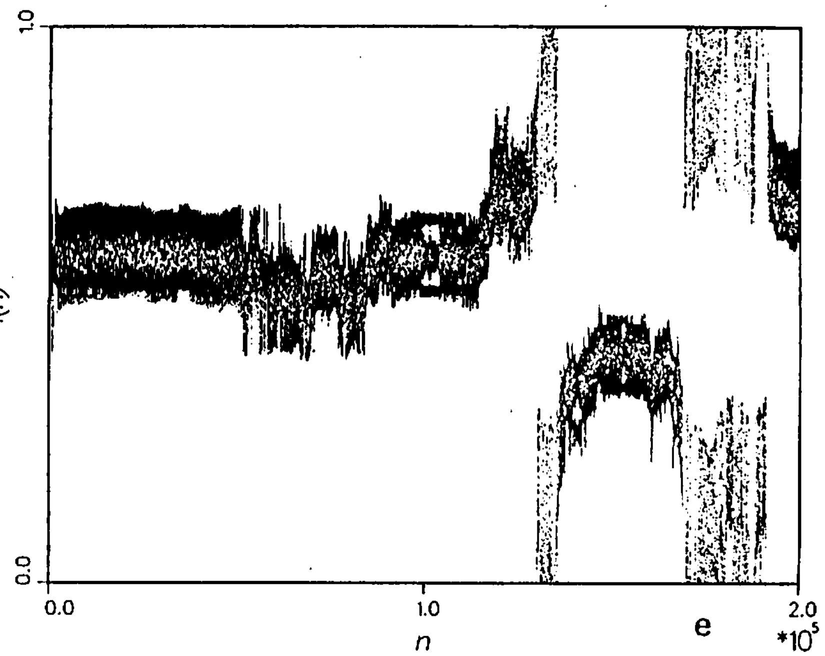


Fig. 1. Iterations of the map (2) for $k_1 = k_2 = 0.8$ and $b = 0.02$ with the initial values $(I, \theta, J, \psi) = (0.5, 0.3, 0.4, 0.2)$ are projected onto the I, θ plane. (I_n, θ_n) are plotted for (a) $1 < n < 5 \times 10^4$, (b) $5 \times 10^4 < n < 10^5$, (c) $10^5 < n < 1.5 \times 10^{15}$, and (d) $1.5 \times 10^5 < n < 2 \times 10^5$. (e) Plots (n, I_n) for the entire run.

happens to hit on the KAM torus. If $b \neq 0.0$, an orbit can cross the remaining KAM tori. Thus, a general feature of the stochastic orbit is the adherence of the orbit close to some KAM tori followed by the escape of the orbit to another stochastic layer by Arnold diffusion.

Next, we will consider the statistical property for the escape from one stochastic layer (near the $\frac{1}{2}$ resonance) to another layer (near the 1 resonance). We choose an ensemble of 625 initial points on a square grid, such that $J_0 = 0.4$, $\psi_0 = 0.2$, $0.52 \leqslant I_0 \leqslant 0.568$, and $0.70 \leqslant \theta_0 \leqslant 0.796$, which are inside of the region surrounded by the last KAM tori for the standard mapping. We take a rectangular region R defined by $0.28 < I < 0.72$ and $0.0 < \theta < 1.0$, and calculate how many points are remaining in R with respect to the time evolution of the initial conditions. The ratio of such surviving points decrease almost exponentially as $e^{-n/\tau}$ and the decay time τ is estimated from the results. τ increases rapidly as b approaches 0.0. The function $\tau(b)$ is fit by:

$$\tau = (c_1/b)\exp(c_2/b),$$

where $c_1 \approx 1.3 \times 10^3$ and $c_2 \approx 1.9 \times 10^{-2}$ for $k_1 = k_2 = 0.8$ (see fig. 2); the data for $k_1 = k_2 = 0.5$ also gives a similar dependence on b. From this data however we cannot exclude the possibility that $\tau \approx b^{-\alpha}$ ($\alpha \approx 3.0$). We at present cannot distinguish between the two functional dependences.

The analytical estimate for $\tau(b)$ is not at hand now. If the system is close to integrable and ϵ is a measure of the strength of the perturbation from the integrable system (e.g. $k = b = 0.0$), some estimates like

$$(1/\epsilon)\exp(1/\epsilon^{\beta})$$

have been obtained by Arnold [7] and Nekhoroshov [10]. Our results are not from the very near integrable regime, but in considering the nonperturbative aspects of Arnold diffusion, similar behavior for τ as a function of the coupling might be expected.

In the case of the standard mapping, the diffusion resulting from the collapse of the KAM torus has recently been investigated [4], where the "cantorus"

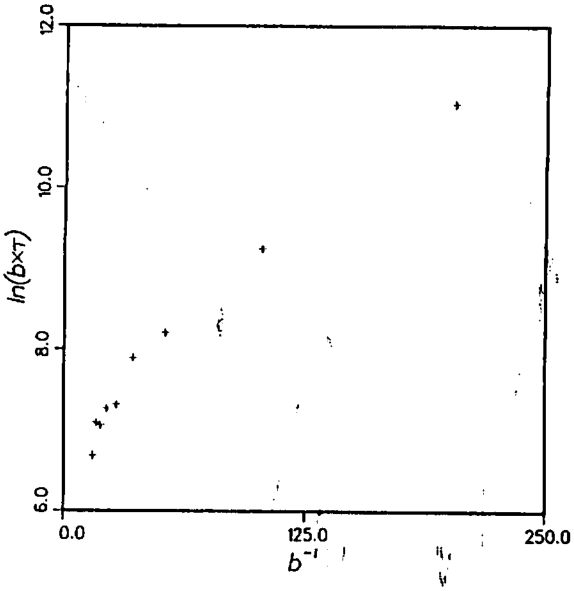


Fig. 2. Dependence of escape time on the coupling b. Log($b\tau$) versus b^{-1} is plotted for $k_1 = k_2 = 0.8$.

determines the limiting process for the diffusion, and the critical behavior for the diffusion rate near $k_l \approx k_c$ has been obtained. In our problems, the limiting process for the diffusion is the density of intact KAM tori. Thus, the mechanism is quite different from transport mediated by "cantori" that may exist in our mapping. Indeed, KAM tori may influence transport in high dimensional systems in a manner similar to "cantori" in systems with only two degrees of freedom.

Determination of the structure of the stochastic layer in phase space is an important problem. In the two-dimensional standard mapping, it is shown that the set of stochastic regions forms a fat fractal for $k > k_c$, in the sense that regions where stochastic orbits are excluded show a scaling behavior [5]. What is the structure of the collection of stochastic orbits in high dimensional systems?

It is believed in generic high dimensional hamiltonian systems [1,7,9] that (1) the set of KAM tori does not form an open set in the phase space though they have a finite measure if the nonlinearity is small, and (2) all stochastic layers which exist in any arbitrary neighborhood of a KAM torus are connected by the so called Arnold web. If both conjectures are true, the closure of the stochastic orbit densely covers the entire energy surface. The Arnold diffusion, however, is extremely slow especially near some special tori, and the whole region cannot be covered within a finite time.

As a method of visualization of the orbit in the four-dimensional phase space, we take a slice of the space in two dimensions, the slice then projected onto the two remaining bases. In fig. 3a, the points $(I_n, \theta_n, J_n, \psi_n)$ which satisfy $0.5 < \theta < 0.51$ and $0.5 < \psi < 0.51$ are plotted. First, we note that there are regions which seem to resist stochastic penetration. The measure of invariant tori in these "holes" is large. However, the area of the vacant regions decreases more slowly with increasing iterations, which leads us to question the relevance of the ergodic theorem within timescales which are less than cosmological. After 4×10^8 iterations, the vacant regions still comprise about 45% of the (I_n, J_n) plane.

As can be seen in fig. 3a, "whiskers" protrude into the vacant regions. Since the torus corresponds to the one as in fig. 3b (which converges to two points as the width of the slice goes to zero), the whiskers can be regarded as a "whiskered torus", which is essential to the Arnold diffusion.

Since the Arnold diffusion is rather slow and the orbit stays for a long time in the vicinity of an invariant torus, some quantities show an intermittent-like behavior. In fig. 4, the short time average value of the action for

$$\tilde{I}_l = K^{-1} \sum_{k=1}^{K} I_{k+(lK)} \qquad (3)$$

is depicted. K is some integer chosen so that it is smaller than the Arnold diffusion time but long enough to represent a statistical average. If the orbit stays at the region near the torus much longer than K steps, it shows a laminar behavior, and when the orbit goes to another stochastic layer it shows a "burst-like" behavior. Thus, the high dimensional hamiltonian system shows a behavior analogous to intermittency in dissipative systems.

Similar behavior also appears for the Lyapunov exponents. If we make short time averages of the Lyapunov exponents and plot them as a function of time, they parallel the behavior of fig. 4. When the orbit stays close to tori, the exponents are almost zero

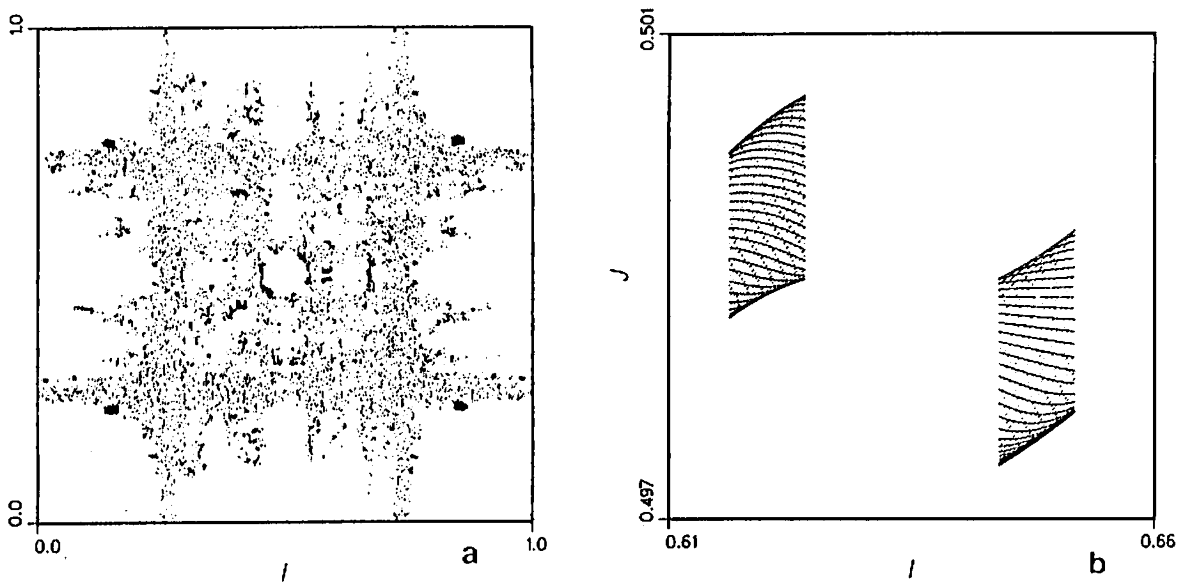


Fig. 3. (a). A representation of a stochastic layer: (I_n, J_n) are plotted, the projection includes points within a finite slice in θ_n and ψ_n, and $1 < n < 4 \times 10^8$. (b). The same procedure is used to illustrate a torus in phase space: $1 \leqslant n \leqslant 4 \times 10^6$.

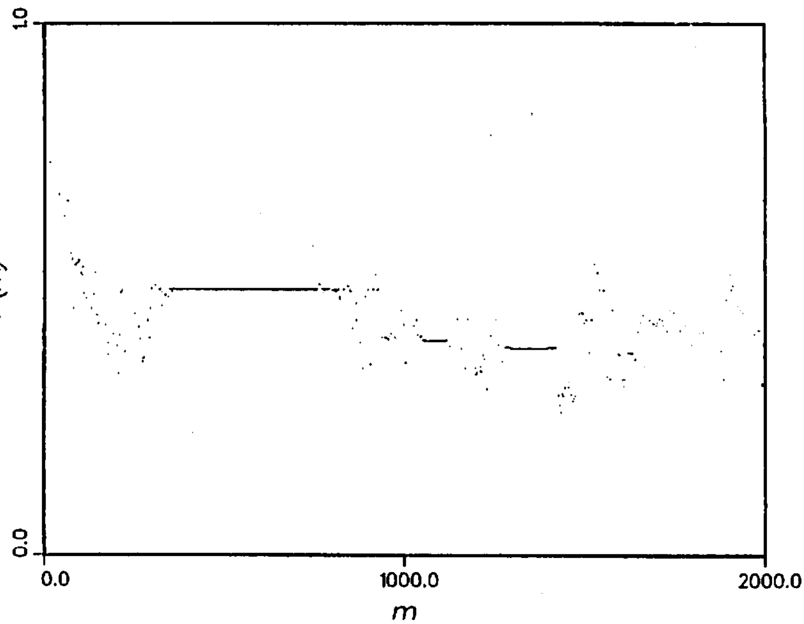


Fig. 4. Averaged action $\tilde{I}$ defined by (3) with $K = 1000$. The parameters for the map are $k_1 = k_2 = 0.5$ and $b = 0.01$ with initial values $(I_0, \theta_0, J_0, \psi_0) = (0.2, 0.5, 0.6, 0.4)$.

Table 1
Lyapunov exponents for $k_1 = k_2 = 0.5$ and $b = 0.02$ with averages over 10^6 iterations.

I	θ	J	ψ	Lyapunov 1	Lyapunov 2	Motion
0.100	0.200	0.300	0.400	4.1×10^{-2}	7.4×10^{-3}	stochastic
0.100	0.400	0.200	0.300	2.6×10^{-2}	5.5×10^{-3}	stochastic
0.500	0.300	0.400	0.200	1.3×10^{-5}	8.4×10^{-6}	quasiperiodic
0.432	0.111	0.353	0.222	1.5×10^{-5}	9.7×10^{-6}	quasiperiodic
0.018	0.019	0.254	0.488	4.7×10^{-2}	9.0×10^{-3}	stochastic

and the first two exponents show a sudden increase when the orbits go out to a region which correlates to a stochastic layer for the original standard mapping. The long time averages give two positive Lyapunov exponents for the stochastic orbit. Some examples of the calculation are shown in table 1.

In summary, the following results have been presented: (1) confirmation of the existence of the Arnold diffusion and its visualization; (2) the observation that a stochastic orbit stays close to a KAM torus, and then escapes to other stochastic layers; (3) dependence of the escape time on the coupling; (4) the structure of the stochastic layer, which we may term a fuzzy fat fractal; and (5) Lyapunov exponents for the stochastic layer.

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