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Transition, Ergodicity and Lyapunov Spectra of Hamiltonian Dynamical Systems

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Hamiltonian systems on a one-dimensional lattice with discrete time are studied. As the coupling constant is increased, they show a sharp transition from regular to random motion. Below the threshold, KAM tori and a stochastic layer coexist. The stochastic motion therein is sticky to KAM tori and has a long time-tail. The motion above the threshold is ergodic, characterized by the power spectra and Lyapunov spectra which are consistent with the results of random matrices.

§1. Introduction and Models

The understanding of the nature of Hamiltonian dynamical systems has progressed rapidly, especially for systems with 2 degrees of freedom, as are predominantly studied by standard mapping.[1] In this way, the mechanism of the collapse of KAM tori, the onset of diffusion, the self-similar structure of islands, and flicker noise are clarified.[2]

The characteristic feature of the Hamiltonian systems with many degrees of freedom has not yet been well understood. For the systems with many degrees of freedom, it is believed that the small-scale structure in the low-dimensional systems is smeared out and the ergodicity is attained even for the weak nonlinearity through the Arnold diffusion.[3-5] The direction of this study stems from the celebrated works by Fermi, Pasta, and Ulam,[6] but subsequent progress has remained rather slow.[11]

Since the time scale involved in the dynamics of many degrees of freedom can be very large, it is desirable to start with a model suitable to numerical simulations. In the present letter, a class of symplectic map lattice systems is investigated in a manner similar to that of the recent coupled map lattice studies for the spatiotemporal chaos.[7] Transition from regular to stochastic motion is reported, along with the Lyapunov analysis and other statistical and dynamical characterizations.

Let us start with the following symplectic dynamical systems:[5,8]

$$\begin{aligned} x_{n+1}(i) &= x_n(i) + p_{n+1}(i), \\ p_{n+1}(i) &= p_n(i) + F^i(\{x_n(j)\}), \end{aligned} \tag{1}$$

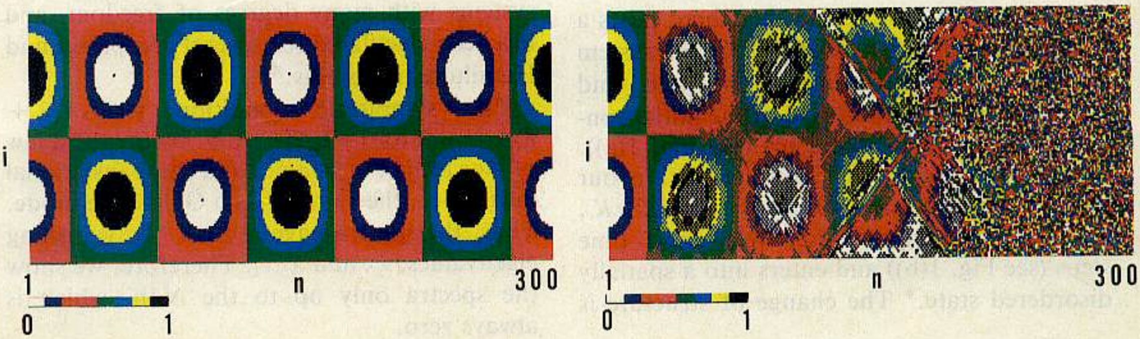

Fig. 1. The space-time color plots for model (4). Spatiotemporal pixel (i, n) is painted according to the color code for $x_n(i)$ mod 1 in the figure. The system size $N=100$, with the initial conditions $x_n(i)=0.5+0.5 \sin(2\pi i/N)$ and $p_n(i)=0$ for all i. (a) $K=0.99$. (b) $K=1.10$.

where the indices $i=1, 2, \cdots, N$ correspond to the spatial lattice site with periodic boundary conditions, and n represents the discrete time step. Here, $x_n(i)$ is the displacement of site i and $p_n(i)$ is its conjugate momentum. The symplectic condition

$$\sum_{i=1}^{N} \mathrm{d}x_{n+1}(i)\wedge \mathrm{d}p_{n+1}(i)=\sum_{i=1}^{N} \mathrm{d}x_n(i)\wedge \mathrm{d}p_n(i),$$

imposes the following restriction on the force term:

$$\partial F^i(\{x(k)\})/\partial x(j)=\partial F^j(\{x(k)\})/\partial x(i). \tag{2}$$

Here, we consider the one-dimensional lattice equation with nearest-neighbor interaction. We further assume that the interaction depends only on the difference between the displacements of the two neighboring sites. Thus, the force term is expressed as

$$F^i=G(x(i+1)-x(i))+G(x(i-1)-x(i)), \tag{3}$$

with

$$G(x)=-G(-x).$$

In this letter, the model

$$G(x)=(K/2\pi)\sin(2\pi x), \tag{4}$$

is studied, but the aspect we discuss is expected to be observed in a large class of dynamical systems with (3).

§2. Stochastic Transition

A remarkable feature in the system (4) is the transition which takes place as the coupling K is changed. Let us take a smooth initial condition such as $x(i)=b+a\sin(2j\pi i/N)$, $p(i)=0$, j=integer. We have found that there exists a critical value K_c (~ 1.0). If $K<K_c$, the system oscillates quasi-periodically in time and returns to the neighborhood of the initial conditions after short time steps (see Fig. 1(a)). The regular oscillation continues within our simulation steps (say, 10^5 steps). For $K>K_c$, the pattern collapses after very short time steps (see Fig. 1(b)) and enters into a spatially disordered state.* The change of structure is characterized by the spatial power spectra. In the regular region, only the discrete delta peaks at nj/N (n: simple integer) are observed, while white spectra without any notable peaks are found in the stochastic region.

The system remains chaotic even at $K<K_c$ if we start from a fully random initial configuration. The point is that the stochastic transition is rather sharp for most smooth initial conditions. For example, the critical value K_c is almost insensitive to the choice of the parameters b, a and j in the sine-wave initial conditions. It is also independent of the system size if the size is large enough ($N>8$). This kind of sharp stochastic transition is also seen in the lattice systems with continuous time, where the transition occurs as the anharmonicity is increased.[9]

§3. Lyapunov Spectra

Next, we investigate the Lyapunov spectra of the model (4). A Lyapunov spectrum is defined as the eigenvalue spectrum $\{\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_{2N}\}$ of a $2N\times 2N$ Jacobi matrix;

$$\lim_{T\to\infty}\frac{1}{T}\log\{\partial(p_T, x_T)/\partial(p_0, x_0)\}$$

$$=\lim_{T\to\infty}\frac{1}{T}\log\left\{\prod_{n=1}^{T}\begin{pmatrix}\mathbf{1} & C_n\\ \mathbf{1} & \mathbf{1}+C_n\end{pmatrix}\right\}, \tag{5a}$$

where C_n is an $N\times N$ matrix whose elements are

$$(C_n)_{ij}=\frac{\partial F^i}{\partial x_n(j)}(\{x_n(k)\}). \tag{5b}$$

It is a (collection of) local expansion rate(s) which is a basic quantity for dynamical systems with many degrees of freedom, and has been calculated for several dissipative and Hamiltonian systems.[10]

The symplectic condition yields $\lambda_i+\lambda_{2N+1-i}=0$ for $i=1, 2, \cdots, N$. By the third law of Newtonian mechanics and the translational invariance, the system has a Goldstone mode. Thus, the spectra have at least two vanishing eigenvalues, λ_N and λ_{N+1}. Therefore, we show the spectra only up to the N-th, which is always zero.

Figure 2 shows Lyapunov spectra for the system (4) with random initial conditions for $K=0.2, 0.4, \cdots, 2.0$. We note that, roughly

* Spiral solutions $x(i)=ij/N+\text{const.}$, $p(i)=\text{const.}$ (j=integer) lose their stability at $K_j=1/\cos(2\pi j/N)$, which accumulate to $K=1.0$ as $N\to\infty$.

creases with K, and is 2 at $K=K_c$. For example, $\alpha=1.34$, 1.57, 1.88, 1.99, 2.00, and 2.00 for $K=0.1$, 0.2, 0.4, 0.6, 0.8, and 1.0, respectively. This kind of flicker-like noise indicates that the motion has a long time-tail and is sticky to KAM tori, as is often observed in the stochastic motion in the standard map.[2)]

§5. Discussion

We have investigated a class of symplectic map lattices. They show a sharp transition from regular to ergodic-like motion as the nonlinearity parameter increases. In the regular regime, both the KAM tori and stochastic motion have a large measure in the phase space. The stochastic motion therein is affected by the KAM tori and has a long time-tail. In the ergodic-like region, the measure of KAM tori is too small to be observed. The Lyapunov spectra are well-approximated by the random matrices, with uniform distribution of x and without spatiotemporal correlation. Thus, the system can be termed "ergodic". The ergodicity is also checked through the temporal power spectra with ω^{-2} and the spatially white power spectra.

The sharp transition from a regular to an ergodic region may be generic in a Hamiltonian system of many degrees of freedom with local interaction; this will be of importance to the understanding of the origin of statistical mechanics.

A detailed study on the symplectic map lattice will be reported elsewhere with an extension to the high-dimensional systems and the inclusion of the other terms in the force.

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Note added—After the completion of the manuscript, we received a preprint "Chaos in Low-dimensional Hamiltonian Maps" by H. Kantz and P. Grassberger (WU B 87-13), where Lyapunov spectra for maps the same as our model are calculated and convex spectra are observed.

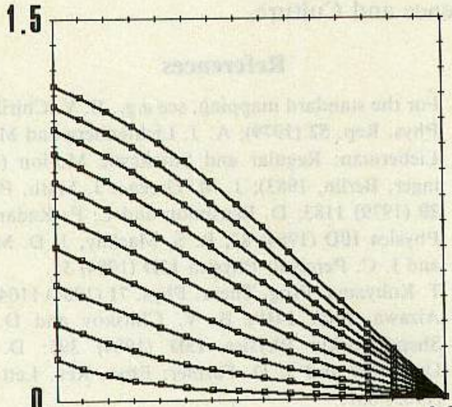

Fig. 2. Lyapunov spectra of model (4). System size $N=16$, initial condition=random, $K=0.2, 0.4, \cdots, 2.0$ (from the bottom of the figure to the top).

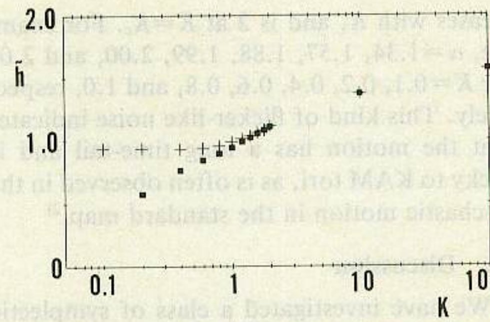

Fig. 3. Scaled KS-entropy $\tilde{h} \stackrel{\leftarrow}{=} \{\sum_{j=1}^{N} \lambda_j\}/(N\lambda_1/2)$. Values of $\tilde{h}$ for model (4) (squares) and for the random matrices (crosses) are shown. System size $N=16$. For the model (4), initial condition=random. Simulation is carried out over $10^5 \sim 5\times10^5$ steps for the model (4) and 2×10^4 steps for the random matrices.

speaking, the spectra are concave for $K<K_c$, while they are convex for $K>K_c$, and linear at $K\sim K_c$. The shape of the scaled Lyapunov spectrum, i.e., λ_i vs i/N, is independent of N for large $N(\gtrsim 8)$.[10,12]

For the region $K \gtrsim K_c$, the spectra are quite similar to those of random Jacobi matrices,[13] which are obtained by replacing x's in the matrices (5) by (spatially and temporally) independent uniform random numbers in the interval [0, 1]. This supports the picture that the region $K\gtrsim K_c$ is a fully developed turbulent regime, where both spatial and temporal correlations decay quite rapidly. On the other hand, the random Jacobi matrix with $K\lesssim K_c$ does not show such a strongly concave spectrum as the actual mapping (4) indicates. Figure 3 shows scaled KS-entropy

$$\tilde{h} \stackrel{\leftarrow}{=} \left\{ \sum_{j=1}^{N} \lambda_j \right\} \bigg/ (N\lambda_1/2). \qquad (6)$$

If the spectrum is linear, then $\lambda_j=(N-j)/(N-1)\lambda_1$ and $\tilde{h}=1.0$. We can see that $\tilde{h}$(map)$=\tilde{h}$(random matrix) for $K\gtrsim K_c$, whereas $\tilde{h}$(map)$<\tilde{h}$(random matrix) for $K\lesssim K_c$. Note that the random Jacobi matrices do not always give linear spectra. The spectra of ergodic regions are characterized not by the linearity as claimed in ref. 14, but by the coincidence with those of the random Jacobi matrices.

In the region with small K, the KAM region and islands take up a large measure in phase space, where all the Lyapunov exponents are zero. The system undergoes a sticky motion around KAM tori, resulting in lengthy temporal correlations, which cannot be reproduced by the random Jacobi matrices. Thus, we interpret that the concavity reflects the existence of large KAM regions and islands.

For the case of the initial condition $x(i) = b + a\sin(2j\pi i/N)$, $p(i)=0$, as we did in the previous section, we have observed Lyapunov exponents to be i) all zero for $K<K_c$, and ii) the same as those in Fig. 2 for $K>K_c$. That is, when $K>K_c$, the Lyapunov spectra are independent of initial conditions. In the regime $K<K_c$, a large KAM region and a stochastic layer coexist.

§4. Characterization of Stochastic Motion

Power spectra in space and/or time are used to characterize the spatiotemporal chaos. The temporal power spectra for $p_n(i)$ are numerically calculated for our model. At $K>K_c$, they give no sharp peaks and are well-fitted by ω^{-2}, which means that the motion of momenta is well-approximated by $p_n(i) = p_{n-1}(i) +$ random white noise. Contrary to the well-known Lorentzian form, the plateau at low frequency cannot be observed. This comes from the translational invariance of our system. At $K<K_c$, the spectra are composed of the ensemble of discrete sharp peaks for the quasi-periodic motion, while for random motion, they contain no sharp peaks and are fitted by $\omega^{-\alpha}$ $(1<\alpha<2)$. The exponent α in-