Complex Dynamical Behaviour in Economic Production Networks

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Abstract

A dynamical model of an economic production network is proposed as an extension of von Neumann's static model. Sets of input commodities are jointly converted into sets of output commodities by production processes. The rate of production of a given production process is limited by the minimum quantity of its set of input commodities. It is shown that economic production is autonomously sustained, without price divergence, even without the imposition of any external constraints on price determination. This is accomplished through switching between different choices of the minimum input commodity, which leads to the appearance of complex cyclic behaviour. The mechanism and origin of this oscillation is analyzed in depth. The generation of complex oscillations with multiple timescales is also shown when several processes are combined to form a chain or a network.

1 Introduction.

We propose a non-linear dynamical model of a production network with the motivation to understand the dynamical origin of non-equilibrium macroeconomic phenomena. The model we propose is a new type of dynamical economic model based on von Neumann's neoclassical model of economic production [von Neumann (1945), Morishima (1970)].

Von Neumann's model is similar to a chemical reaction in some respects. He regarded an economic production process as a transformation from one set of commodities to another set of commodities. A set of input commodities is

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jointly converted to another set of commodities as production output. This transformation process, which is termed "joint production" in the economics literature, is similar to a chemical reaction that transforms several chemicals to other chemicals. Generally, chemical reaction equations have multiple elements on both sides, as in $n_1A_1 + n_2A_2 + \cdots + n_kA_k \to m_1B_1 + m_2B_2 + \cdots + m_\ell B_\ell$, for example. Here multiple chemicals, jointly, are transformed into multiple output chemicals, as in economic joint production. In such chemical reactions, there may be some chemicals which are necessary but are not transformed, i.e., the same term may appear on both sides of the reaction equation. This is nothing but a catalyst. Indeed, in von Neumann's treatment of production, some commodities play the role of catalysts. Some machines are necessary for production but are not transformed by the production process. These are similar to catalysts in chemical reactions. Finally, von Neumann describes economic production as a network consisting of several production processes. In this sense his description of an economic production network is similar to a chemical (catalytic) reaction network.

In the present paper, we will introduce a dynamical version of von Neumann's model, noting the above analogy with a chemical reaction network. In fact, applications of catalytic reaction network dynamics to biological systems, as pioneered by Alan Turing (1952), have recently been developed for biological cell production systems by Kaneko *et al.* (1997), and Furusawa *et al.* (2001). We introduce our dynamical von Neumann model with reference to this description of biological cells in terms of reaction networks.

Here, we also note that Padgett *et al.* (2003) constructed an agent based economic model, inspired by the theory of the hypercycle in a chemical reaction network. They studied the emergence of a complex production network structure amongst agents with simple decision making rules, while the dynamics on the network were not investigated.

Some economic theorists have studied the origin of economic instability, with regard to economic dynamics. Several models of "endogenous business cycles" have been proposed, for example Goodwin (1951, 1967), Kaldor (1940) and Kalecki (1935, 1937, 1939, 1954). These models had many followers like Goodwin, Kruger and Vercelli (1984), Goodwin and Punzo (1987, 1989), Dore (1993), Tvede (1997) and other works referred to in these books. However, these models are all low-dimensional macro-economic models and did not refer to the instability originating from the microscopic-level interaction of several production processes.

Morishima (1970) discussed the growth path of the von Neumann model and showed that the path can follow a zig-zag which does not follow the von Neumann steady-growth path. Goodwin (1989) discussed a multi-sectoral model without joint production where each sector has its own limit cycle with a different periodicity. He showed that the growth path fluctuates around the von Neumann steady-growth path. However, these works just indicated the possible complex behaviour of this type of model. They did not study the richness of the behaviour.

In contrast, we propose a novel framework for understanding economic fluctuations whereby such fluctuations originate in the instability which arises from interactions of production processes, in a similar way to the dynamical instability of chemical reaction networks. Even a simple toy model, where a production process involves just three commodities, shows non-trivial oscillatory behavior where switching between minimum commodities is observed. When we consider a network describing the production of multiple commodities, complex cyclic behaviour appears with alternation between multiple time scales.

This paper is organized as follows; In §2, we briefly review the original von Neumann model of an economic production network. Inspired by it, we propose our dynamical model in §3. Numerical results of the model are given in §4, where non-divergent economic production processing is shown to be generated autonomously through oscillatory dynamics. The mechanism for this is analysed and shown to be due to switching over choices of distinct minimum terms. By studying a chain of production processes and also a network of production processes, complex oscillatory dynamics with multiple time scales are observed. The relevance of our model and results to economic dynamics is discussed in §5, while the universality of our results is discussed in relation to several extensions of the original model.

2 The original von Neumann Model

John von Neumann proposed a multi-sectional economic growth model [Neumann (1945)].

The important ideas of his model are as follows: (i) joint production, (ii) the inequality approach, (iii) dual-cross adjustment, and (iv) the time-lag of production. Hereafter we will briefly explain the original von Neumann model according to Morishima (1970).

Let us suppose there are N kinds of commodities and the economy has M kinds of production process. A production process refers to a method for transforming some commodities into other commodities. For example:

> (1 egg, 1 new pan, 1 cooking stove, a little salt, a little pepper, 3 minutes labour) $\rightarrow$ (1 fried egg, 1 slightly older pan, 1 slightly older cooking stove.)

The left hand side is the "inputs" and the right hand side the "outputs". There are several goods on the right side, and this reflects von Neumann idea of introducing joint production.

Amongst economists, the joint production function introduced by Sraffa (1960) is more famous than von Neumann's. However, Sraffa's publication is far later than von Neumann's (1945). In addition, Sraffa's "equation" approach is less useful than von Neumann's inequality approach. The famous Leontief model does not include joint production, and is merely a special case of this von Neumann model [Leontief (1941)].

The first production process can be described as follows;

$$(a_{11}, a_{12}, ..., a_{1N}) \rightarrow (b_{11}, b_{12}, ..., b_{1N}).$$

This means that the first process consumes a quantity a_{11} of commodity 1, a quantity a_{12} of commodity 2, ..., and a quantity a_{1N} of commodity N. It also produces a quantity b_{11} of commodity 1, a quantity b_{12} of commodity 2, ..., and a quantity b_{1N} of commodity N. If commodity k is not an input in this process, $a_{1k} = 0$, while if this commodity is not an output, $b_{1k} = 0$. In general the l-th production process is described as

$$(a_{l1}, ..., a_{lN}) \rightarrow (b_{l1}, ..., b_{lN}).$$

This kind of production function has an L-shaped isoquant. The set of production processes can be described by an $M \times N$ input matrix $A = (a_{ij})$ and an $M \times N$ output matrix $B = (b_{ij})$. The matrices A and B are fixed, since von Neumann assumes that there is (i) no innovation and (ii) constant returns to scale.

The price of each commodity is defined as the column vector, $p = (p_1, p_2, ..., p_N)'$, and the activity level (or intensity) of each process as the row vector $z = (z_1, z_2, ..., z_M)$. The total input vector of commodities of this economy is then,

$$zA = (\sum_{i=1}^{m} z_i a_{i1}, \dots, \sum_{i=1}^{m} z_i a_{iN}),$$

and the total output vector is

$$zB = (\sum_{i=1}^{m} z_i b_{i1}, \dots, \sum_{i=1}^{m} z_i b_{iN}).$$

In his model von Neumann assumes a time-lag of production. This means that input commodities are converted into output commodities in all processes simultaneously in fixed non-overlapping sequential time periods. Therefore the quantities of each commodity available for input each period cannot exceed the quantities output in the previous period. This means the relationship between supply and demand is fixed according to the production output in the preceeding period and the input in the current period. We introduce t which denotes the time period and write the price vector p and intensity vector z of period t as $p(t)$ and $z(t)$. The relationship between supply and demand is written as

$$z(t)B \geq z(t+1)A. \tag{1}$$

We now write the rate of profit of production process i as $\Pi_i(t)$ given by,

$$\Pi_i(t) = \frac{\{\sum_{j=1}^{N} b_{ij}p_j(t+1) - \sum_{j=1}^{N} a_{ij}p(t)\}}{\sum_{j=1}^{N} a_{ij}p_j(t)},$$

the highest rate of profit $\Pi(t)$ amongst $\Pi_i(t)$ $(i = 1, ..., M)$ then fulfills,

$$\Pi(t) \geq \frac{\{\sum_{j=1}^{N} b_{ij}p_j(t+1) - \sum_{j=1}^{N} a_{ij}p(t)\}}{\sum_{j=1}^{N} a_{ij}p_j(t)}.$$

If we write $\beta(t) = 1 + \Pi(t)$, we have

$$Bp(t+1) \leq \beta(t)Ap(t). \tag{2}$$

According to the idea of dual cross adjustment, the prices of the commodities affect the supply-demand relationships, while the supply-demand relationships affect the prices of the commodities. This can be realised by two neo-classical rules; (i) the free goods rule, and (ii) the profitability rule.

The free goods rule: If the supply of a commodity exceeds its demand at equilibrium, its price is set to 0. That is, if the strict inequality '>' applies in (1) for j in equilibrium, then $p_j = 0$. Therefore, we have

$$z(t)Bp(t+1) = z(t+1)Ap(t+1). \tag{3}$$

The profitability rule: If production process i has a rate of profit which is less than the highest rate of profit, no entrepreneur will use this process. That is, if the strict inequality '<' applies in (2) for production process i in equilibrium, $z_i(t) = 0$. Therefore, we have

$$z(t)Bp(t+1) = \beta(t)z(t)Ap(t) \tag{4}$$

Von-Neumann constructed a system with these four equations and discussed the path of economic development. He assumed that labour is freely adjusted to its demand, and that its price (the wage) is constant. According this assumption, wages are kept constant and labour supply and demand are not included in the equations.

The problem which von Neumann addressed with this system concerned the existence of a long term equilibrium growth path. In this situation the activity levels (or intensities) of all production processes increase (or decrease) from period to period at the same rate, and all the prices are constant. In this case, the rate of increase in activity, α is constant in all periods. Namely, $z(t+1) = \alpha z(t)$ for all t. Because all prices are constant, all rates of profits are constant. Thus $\Pi(t) = \Pi$ and $\beta(t) = \beta$. This means that the quantities, $x_i = z_i(t)/\sum_p z_p(t)$ are constants, as are the quantities $y_j = p_j(t)/\sum_q p_q(t)$.

Von-Neumann demonstrated the existence of a long term equilibrium development path under the following assumptions;

$$A \geq 0, \;\; B \geq 0, \tag{5}$$

$$A + B > 0. \tag{6}$$

However, the economic meaning of the assumption (6) is that every commodity is involved (either as an input or an output) in every process. Needless to say, this assumption is too strong. For example, an apple is not involved as an input or an output in the production process of computer.

In addition, von Neumann neglected the important condition that $xBy > 0$, which is necessary for the economy to be meaningful. If this condition is broken, not all commodities are produced, or all prices are 0. Kemmeny *et al* (1956) added this condition and replaced assumption (6) with the following reasonable conditions: "Every production process consumes at least one commodity" and "Every commodity is produced by at least one production process." They demonstrated the existence of a long term equilibrium development path under these more reasonable conditions.

As we have seen, the original von Neumann model addressed only the long run equilibrium of economic growth. Morishima (1970, 1992) tried to revise this model to include dynamics. His approach was to reconstruct the von-Neuman model according to the Hicksian paradigm of a series of temporary equilibriums [Hicks (1939)]. In this framework, Morishima addressed (i) the Turnpike Theorem, (ii) the Optimal Growth Path, (iii) Monetary Instability, (iv) Innovation by Entrepreneurs, and other topics. However, his treatment was purely analytical. He did not perform any numerical simulations. It is obvious that a time series of temporary equilibriums will not be identical to the dynamics that emerge in the disequilibrium state, and for understanding this numerical simulation is a powerful tool.

3 Dynamic von Neumann Model

In order to discuss the disequilibrium dynamics of the von Neumann system, we must remove the two neoclassical rules, namely the free goods rule and the profitability rule. Although these rules are useful for equilibrium analysis, they are not appropriate when discussing disequilibrium. The reason is that in disequilibrium, which processes are profitable and which are not will vary with time, and indeed will depend on the environment of the process. In particular profitability will depend on the state of the other processes in the system, as well as on the external environment. This will be true even without technological innovation, when the interaction matrices are fixed. In the same way it is well known that the concentrations of chemicals in a cell will not necessarily be at a steady state, but may deterministically fluctuate, even chaotically, even though they react in fixed ratios and the reaction matrix is fixed. Another well known example is provided by the populations of species in an ecosystem. Such systems, which can be described by the famous Lotka-Volterra equations, are well known to show permanent oscillations in predator-prey populations, even though the matrix which describes their interactions is fixed.

Besides this, another problem, is the assumption of perfect rationality and perfect foresight of the entrepreneurs. In fact entrepreneurs may not know which processess are profitable when they come to invest in them, and may only discover later when their produced output products are sold on the market. Indeed the production process itself will affect the rest of the system itself in complex and possibly unpredictable ways, which entrepreneurs may not even in principle be able to account for.

In this model we take a more bottom-up view. We make a model where the processes themselves dynamically adapt to their present environment and dynamically affect their environment too. Processes respond in real time to changing external supply and demand of the commodities which they consume or produce. Rather than imposing macroscopic global rules, such as the rule of free goods and the rule of profitability, such rules will be an emergent property of the microscopic interactions between processes. In this model we hope to motivate an economy as a dynamical co-adaptive network of production processes.

However removing these useful rules, we face a problem. In the von Neumann model at the end of a production period t we have an N-dimensional row vector $z(t)B = (\sum_{i=1}^{M} z_i(t)b_{i1}, \ldots, \sum_{i=1}^{M} z_i b_{iN})$, which describes the quantities of the N commodities available at that time. In the original von Neumann model, the two rules are used to fix the M-dimensional column vector $z(t+1)$ describing the process intensities in the next time period. Since we have removed these two rules, we must invent some suitable methods to allocate the available commodity supplies among the processes which need them and therefore fix the intensities of the processes. We do this in perhaps the simplest way possible.

In our model we denote by $S_j(t)$, the total quantity of commodity j available in the economy at time t. We also assume each process i has a value, $F_i(t)$, which we refer to as *funds*. This is the revenue of sales of its produced commodities in the preceeding timestep and therefore reflects the value (or *fitness*) of the process in the economy at that time. We will explain the details below, but each process uses its funds $F_i(t)$ to obtain quantities of the commodities it needs as input material for its processing. We denote these obtained quantities of process i of product j at time t as $S_{ij}(t)$.

The process activity, which we now denote $M_i(t)$, is considered a *rate* of processing. In one production period, $\Delta t = 1$, a quantity $M_i(t)$ of processing occurs in process i. Therefore preceeding a period of processing each process i has quantities $S_{ij}(t)$ of commodity j, while after the period of processing these quantities have become $S_{ij}(t) + (b_{ij} - a_{ij})M_i(t)$.

The funds $F_i(t+1)$ is the proceeds of sales of these commodities produced in the previous time t, so that,

$$F_i(t+1) = \sum_j (S_{ij}(t) + (b_{ij} - a_{ij})M_i(t))p_j(t+1) \tag{7}$$

where $p_j(t+1)$ is the relevant price of product j. Note that this equation includes the quantities of product j produced, i.e. after processing.

The total supply of commodity j, $S_j(t+1)$, available in the whole economy after processing is then given by $S_j(t) + \sum_i (b_{ij} - a_{ij})M_i(t)$, where $S_j(t) = \sum_i S_{ij}(t)$. As explained, we must introduce a procedure to allocate this available supply between the processes which require it. In order to do this we assume that each process uses its funds available from the previous period of processing, $F_i(t)$, to form *demands* for product j by dividing these funds according to the ratios, a_{ij}, that is according to the ratios it needs for processing. Therefore process i forms demands $F_i(t)a_{ij}/\sum_j a_{ij}$, for product j. If we consider the a_{ij} normalized, $\sum_j a_{ij} = 1$, the demands simplify to $F_i(t)a_{ij}$. New supplies $S_{ij}(t+1)$ are then obtained as,

$$S_{ij}(t+1) = \frac{a_{ij}F_i(t)}{p_j(t+1)}. \tag{8}$$

In order for (7) and (8) to hold consistently we require the price of commodity j, $p_j(t+1)$ formed in the economy by this exchange of commodities to be given simply by,

$$p_j(t+1) = \frac{\sum_i a_{ij} F_i(t)}{S_j(t) + \sum_i (b_{ij} - a_{ij}) M_i(t)} \tag{9}$$

i.e. as total demand for commodity j divided by total supply of commodity j. If this is the case, the total supply of commodity j, $S_j(t) + \sum_i (b_{ij} - a_{ij}) M_i(t)$, will be conserved in the commodity allocation and the total funds $\sum_i F_i(t)$ available in the whole economy will also be conserved.

We note that in considering the process i demand $F_i(t)a_{ij}$, for product j, the price of product j does not enter the formular. It is not difficult to incorporate knowledge of the last known price $p_j(t)$ in the demand forming equation, so that the process i demands for product j are given by, $F_i(t)a_{ij}p_j(t)/\sum_j a_{ij}p_j(t)$, rather than $F_i(t)a_{ij}$, but as a first approximation we consider the latter simpler prescription, and defer consideration of the former prescription until the discussion section and later papers.

What about the process activity $M_i(t)$, the amount of processing? The amount of processing is always proportional to the minimum quantity of commodity the process has, i.e.,

$$M_i(t) = Min(\frac{S_{i1}(t)}{a_{i1}}, \frac{S_{i2}(t)}{a_{i2}}, ..., \frac{S_{iN}(t)}{a_{iN}}). \tag{10}$$

where $Min(x_i)$ denotes the minimum over the quantities x_i. For example suppose that at the start of a production period of, say, 1 hour, a production process receives 12 wheels, 3 engines and 2 gear shift sticks. Then it can make no more than 2 cars in that hour, no matter how much machinery and labour it has. Suppose further, for example, that the production process has only one wheel fitting machine, which can only work at a given rate of fitting four wheels an hour. Then even with the above sufficient quantities, the production process can still only make one car in the hour. If there are 5 wheel fitting machines, the process can now make 2 cars in the hour, with the above given supply rates. Clearly the amount of production that can occur in any given time period is directly proportional to the minimum quantity of the process.

The set of equations (7) and (8) with the subsitution of (9) and (10) define the temporal evolution of our model, and using them we calculate the development path of the economy.

4 Results

We have carried out extensive simulations of the model, taking a variety of networks, and we have commonly observed complex oscillatory dynamics. In general, we expect complex macroeconomic behaviour to emerge when many simple microeconomic processes are coupled so that they can adapt to each other. To understand such a system we first study the dynamics of a single process in a *fixed* environment, without other *coadaptive* processes. While such a fixed environment system may seem unrealistic it can even itself be considered a *first approximation* to a complete production economy, as we will describe.

With this in mind we first consider a simple single process example and then we consider a network of such production processes, illustrating the appearance of a multiple timescale limit cycle causing complex unpredictable production dynamics.

4.1 Single Process in Fixed Environment

To illustrate the meaning of this model and its dynamical behaviour we first consider the simplest (non-trivial) example possible. This is a one input product I, one pure catalyst product C, one output product O system in a fixed external environment.

Indeed, as noted, such a system may represent a highly idealized single economy, where the input product is imported at a fixed rate from the external environment and the output product is demanded by the external environment at a fixed rate. Clearly this is an idealization. Nevertheless we believe it is still a relevant system to study and indeed captures the minimal essential characteristics of such a situation. Here we show that this simple system shows complex dynamics, especially given the fact the environment is fixed. In particular we show that the characteristic dynamical behaviour is a limit cycle oscillation, which is a type of threshold oscillation produced by a feedback of the process onto the prices of its supply products and that this in turn causes endogenous sudden price bubbles and crashes. Furthermore we show that the production process shows intuitively correct dynamical behaviour. Even in this simple model, we will show that the supplies oscillate.

The system is described by the input supply $S_I(t)$, the catalyst supply $S_C(t)$ and the process value $F(t)$. We consider the process perfectly efficient and set the production coefficients as $a_I = a_C = b_C = b_O = 1/2$ and $b_I = a_O = 0$. We now assume that all the other processes and the external environment is fixed and describe it by a fixed set of environment parameters. They are S_I^{ext}, S_C^{ext}, and S_O^{ext} describing fixed external supply rates of the input, catalyst

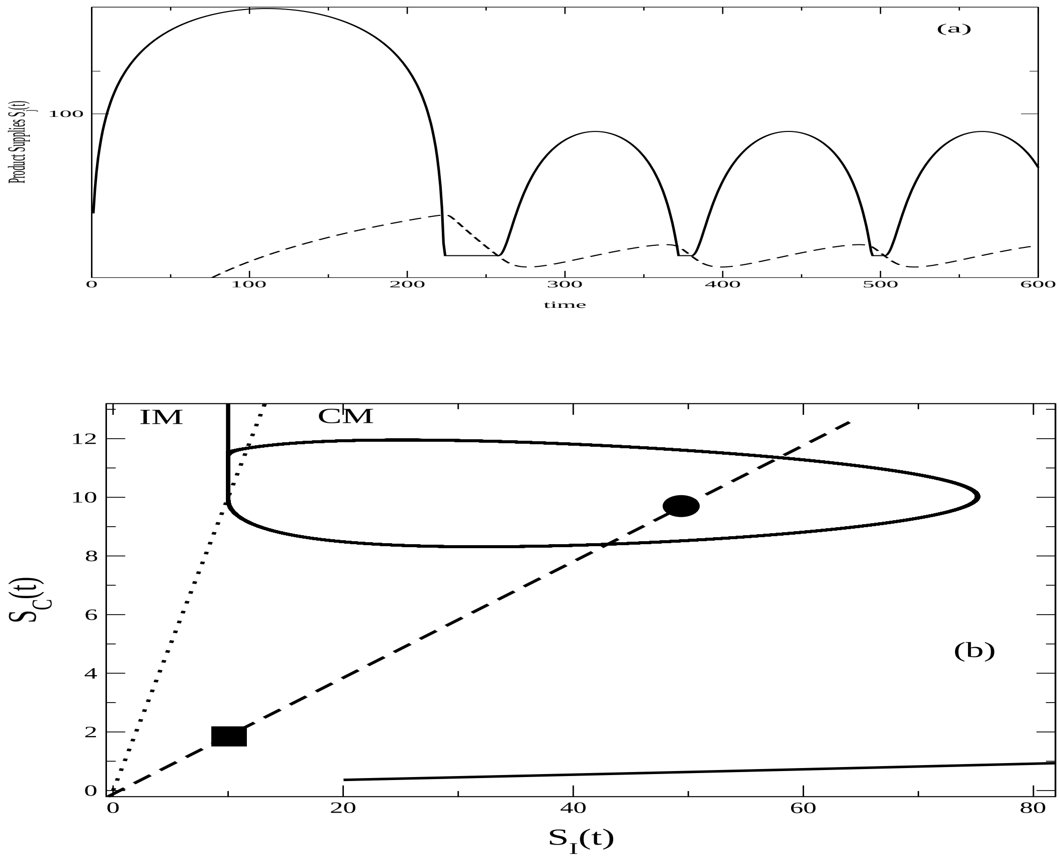

Fig. 1. (a) Single process supplies time series, $S_I(t)$ solid, $S_C(t)$ dashed. (b) The same time series shown in the $S_I(t)$, $S_C(t)$ plane. The dotted line is the $S_I(t) = S_C(t)$ line and divides the plane into the (CM) and (IM) regions, as indicated. When the trajectory crosses this line the equations *switch* the minimum condition. The (IM) fixed point (S_I^{IM}, S_C^{IM}) is shown as a square and the (CM) fixed point (S_I^{CM}, S_C^{CM}) is shown as a circle. They lie on the dashed line of slope $1/\rho$ through the origin. The parameters are $S_I^{ext} = 10$, $D_O^{ext} = 100$, $S_C^{ext} = 0.1$, $D_C^{ext} = 1.5$ and $D_I^{ext} = S_O^{ext} = 0$. $\alpha = 1$, $a_I = a_C = b_C = b_O = 1/2$.

and output, respectively, and D_I^{ext}, D_C^{ext}, and D_O^{ext} describing fixed external demand rates of the input, catalyst and output, respectively.

Now the single process model is described by,

$$
\begin{aligned}
S_I(t+1) &= 1/2F(t)\frac{S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext}}{1/2F(t) + D_I^{ext}}, \qquad (11)\\
S_C(t+1) &= 1/2F(t)\frac{S_C(t) + S_C^{ext}}{1/2F(t) + D_C^{ext}},\\
F(t+1) &= \frac{S_I(t) - M(S_I(t), S_C(t))}{S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext}}(1/2F(t) + D_I^{ext}),\\
&\quad + \frac{S_C(t)}{S_C(t) + S_C^{ext}}(1/2F(t) + D_C^{ext}) + \frac{M(S_I(t), S_C(t))}{M(S_I(t), S_C(t)) + S_O^{ext}} D_O^{ext},
\end{aligned}
$$

where $M(S_I(t), S_C(t))$ is the production function, i.e. the minimum of the catalyst and input supplies.

First we show an example of the time series behaviour of the supplies $S_I(t)$ and $S_C(t)$ in this case in Fig.1. Oscillatory behavior, as shown in this figure, is commonly observed and we will explain how this oscillation arises.

In order to understand this, we consider at first the case where there is no external demand for input product or catalyst, i.e. $D_I^{ext} = D_C^{ext} = 0$. This means the process has no competition when buying its input and catalyst products. Then the process funds $F(t)$ must become irrelevant since the process receives all the available supply anyway. Indeed the supplies equations separate from the funds equation as they should and we get for the supplies equations,

$$\begin{aligned}\Delta S_I(t) &= S_I^{ext} - M(S_I(t), S_C(t)),\\ \Delta S_C(t) &= S_C^{ext},\end{aligned}$$

where $\Delta S_I(t) = S_I(t+1) - S_I(t)$ and likewise for $\Delta S_C(t)$. From these equations it is clear that S_I^{ext} and S_C^{ext} are simply to be thought of as *rates* of external supply of the products. Furthermore the production function M is a *rate* of production. The input supply $S_I(t)$ increases at a rate S_I^{ext} and is consumed at a rate $M(S_I(t), S_C(t))$. While the catalyst supply simply grows at a rate S_C^{ext}. Supposing initially that the catalyst supply is the minimum, $M(S_I(t), S_C(t)) = S_C(t)$, these equations are easily solved so that,

$$\begin{aligned}S_I(t) &= S_I(0) + S_I^{ext}t - S_C(0)t + \frac{1}{2}S_C^{ext}t^2,\\ S_C(t) &= S_C(0) + S_C^{ext}t.\end{aligned}$$

Therefore the catalyst supply $S_C(t)$ grows linearly and the input supply $S_I(t)$ has a quadratic behaviour with a peak. Of course due to this quadratic behaviour $S_I(t)$ will eventually decrease. Eventually these equations will cease to hold since the minimum processing condition $M(t)$ will *switch* when $S_C(t) > S_I(t)$. In this case the appropriate equations become,

$$\begin{aligned}\Delta S_I(t) &= S_I^{ext} - S_I(t),\\ \Delta S_C(t) &= S_C^{ext},\end{aligned}$$

so that $S_C(t)$ will continue to grow linearly but now $S_I(t) \sim S_I^{ext}$. Indeed since there is no competition for the catalyst i.e. $D_C^{ext} = 0$, $S_C(t)$ continues to grow, but $S_I(t)$ is fixed, so there is no more minimum switching and this is the final state with production occurring at rate $M(t) = S_I^{ext}$.

Now let us consider the case that all the external supplies are zero in (11), i.e. $S_I^{ext} = S_C^{ext} = S_O^{ext} = 0$. Now the process is the sole supplier of these 3 products and has no competition to supply them. (This is not possible in practice of course since the process would soon consume all its input product $S_I(t)$ without external supply S_I^{ext}.) Then, again the funds equation decouples from the supplies equation and (11) simply becomes,

$$\Delta F(t) = D_I^{ext} + D_C^{ext} + D_O^{ext}$$

and it is clear that the external demands, D_O^{ext} etc, are also to be considered *rates* of supply of money from the external environment. Indeed the process receives all the available funds at these rates since it is the sole supplier of these products and has no competition to supply them.

We now return to the more general case described by (11), where none of the external supplies and demands are necessarily zero. The initial part, up to $t \sim 270$, is a transient and the roughly quadratic behaviour of $S_I(t)$ and linear growth of $S_C(t)$, as described for the simplified case above, is easy to see from Fig.1(a). The transient can also be seen in the lower right part of Fig.1(b). It is also seen that once $S_I(t)$ has crossed $S_C(t)$ at $t \sim 220$ the input supply goes quickly to a fixed level as it did in the simple case described above, but rather than $S_C(t)$ continuing to grow linearly it now decays. After some time $S_C(t)$ crosses back through $S_I(t)$ and the system switches back to the roughly quadratic $S_I(t)$ growth behaviour. This state is repeated to produce a *permanent switching* state, which is a novel type of limit cycle.

Indeed the origin of this limit cycle is quite novel in dynamical systems theory since it is produced basically by the presence of the minimum condition which switches between the catalyst minimum phase and the input minimum phase. In fact the minimum condition turns a focus fixed point into a limit cycle because the minimum switching boundary “traps” the oscillation around the focus. As we will now describe, this limit cycle can be understood simply by considering the fixed points of the system (11).

4.2 Equilibrium Fixed Points in the Single Process.

We denote the two phases as (CM) for catalyst minimum i.e. $S_I(t) > S_C(t)$ and (IM) for input minimum $S_C(t) > S_I(t)$. In both (CM) and (IM) phases the system has two fixed points. In both cases (CM) and (IM) one of the fixed points is at the origin $(S_I = 0, S_C = 0, F = 0)$, (we will say more about this later.) The other fixed point is at a different position in each phase however. Denoting the (CM) catalyst minimum phase phase fixed point as $(S_I^{CM}, S_C^{CM}, F^{CM})$ and the (IM) input minimum phase fixed point as $(S_I^{IM}, S_C^{IM}, F^{IM})$ we find,

$$S_C^{CM} = S_I^{IM} = \frac{S_I^{ext} D_O^{ext} - S_O^{ext} D_I^{ext}}{D_O^{ext} + D_I^{ext}} = M^{equil} = S_O^{equil}, \tag{12}$$

$$\rho = \frac{S_I^{CM}}{S_C^{CM}} = \frac{S_I^{IM}}{S_C^{IM}} = \frac{D_C^{ext}(S_I^{ext} + S_O^{ext})}{S_C^{ext}(D_I^{ext} + D_O^{ext})}, \tag{13}$$

and

$$F^{CM} = 2S_C^{CM} \frac{D_C^{ext}}{S_C^{ext}} \qquad F^{IM} = 2S_C^{IM} \frac{D_C^{ext}}{S_C^{ext}}. \tag{14}$$

Note that $S_C^{CM} = S_I^{IM}$. Since this is the fixed point of the minimum supply in the two phases, the fixed point of the minimum is the same in both phases and we denote it M^{equil}. This is the equilibrium production and is therefore also the fixed point of the output supply which is also the same in both phases S_O^{equil}. Also see that the ratio of the fixed point values which we denote ρ, is the same in both cases. This geometry means the fixed points lie on a single line of slope ρ through the origin as shown in Fig.1. In fact the variation of ρ controls a novel bifurcation from a non-oscillating state to a limit cycle.

Indeed we notice that at the parameter settings used for Fig.1 while the position of the (CM) catalyst minimum fixed point is inside the (CM) phase, the (IM) fixed point is not inside the (IM) phase. That is $S_C^{CM} < S_I^{CM}$ but $S_I^{IM} > S_C^{IM}$. I.e. the (CM) fixed point is *compatible* with its phase while the (IM) fixed point is *incompatible*. This means that if the trajectory enters the (IM) phase it will be attracted to the (IM) fixed point, but this is actually inside the (CM) phase. Therefore this will cause the system to switch back into the (CM) phase some time later. The other point to notice is that the (CM) phase has an oscillatory character. In fact, although we have not mathematically analysed it, we deduce from observing simulations that this fixed point is a focus, while the (IM) fixed point is a node. The trajectory transiently oscillates around the (CM) fixed point, but before it can decay to it, it crosses back into the (IM) phase producing the novel limit cycle. Note also that if the external environment parameters are such that $\rho < 1$, the system can remain at the node (IM) fixed point without oscillating. In this case $S_I(t)$ and $S_C(t)$ decay to fixed values, with $S_C(t)$ pemanently in excess. At $\rho = 1$ the compatibility state of the fixed points is reversed and therefore $\rho = 1$ produces a novel bifurcation from fixed point to limit cycle. We call this a *trapping* bifurcation.

Furthermore the bifurcation parameter ρ also controls the frequency of the oscillation. This is due to the interaction of the line of slope ρ which contains the fixed points and the minimum switching line $S_I = S_C$. This can be seen from Fig.1(c). As the parameter ρ decreases towards the bifurcation point 1, the frequency of the oscillation will increase and the amplitude of the oscillation will decrease since the focus fixed point will approach the switching boundary.

Indeed numerical simulations confirmed that the switching frequency is rougly linear in $1/\rho$ as would be expected from geometrical considerations.

If we substitute the fixed point values in the two phases (CM) and (IM) equations (12),(13),(14) into the price equations (9) we get the equilibrium fixed point prices. These turn out to be the same in both minimum phases and are given by,

$$p_I^{equil} = p_O^{equil} = \frac{D_I^{ext} + D_O^{ext}}{S_I^{ext} + S_O^{ext}} \qquad p_C^{equil} = \frac{D_C^{ext}}{S_C^{ext}}, \tag{15}$$

where p_I^{equil}, p_O^{equil} and p_C^{equil} are the equilibrium prices of the input, output and catalyst respectively.

The parameter ρ can also be seen to be the ratio of the equilibrium prices,

$$\rho = \frac{p_C^{equil}}{p_I^{equil}} = \frac{p_C^{equil}}{p_O^{equil}} \tag{16}$$

This means the important factor in determining the frequency of the limit cycle oscillation, and indeed whether limit cycle oscillations occur at all, can be seen to be the ratio of the equilibrium prices p_I^{equil} and p_C^{equil}.

Above we also mentioned that each phase has another fixed point, at the origin, the *bankrupt* process fixed point. In this case, or when the process does not exist, the equilibrium fixed point prices are simply given by,

$$p_I^{brupt} = \frac{D_I^{ext}}{S_I^{ext}} \qquad p_O^{brupt} = \frac{D_O^{ext}}{S_O^{ext}} \qquad p_C^{brupt} = \frac{D_C^{ext}}{S_C^{ext}}, \tag{17}$$

where ^{brupt} refers to bankrupt fixed point prices. In fact the interior fixed point, (12),(13),(14),(15), is stable, and the bankrupt fixed point unstable, when $\frac{D_I^{ext}}{S_I^{ext}} < \frac{D_O^{ext}}{S_O^{ext}}$, i.e. when $p_I^{brupt} < p_O^{brupt}$. In other words, when the process gives rise to profit, the process can exist without going bankrupt and it does so by processing input product into output product at a rate such that the equilibrium prices given by (15) hold and the equilibrium price of the input is now equal to the output. Of course the system is not necessarily at a fixed point, but nevertheless will oscillate around it, so that the relations (15) approximately hold on average. On the other hand, if $p_I^{brupt} > p_O^{brupt}$, the process 'goes bankrupt' and does not affect the prices.

Indeed this bifurcation to bankruptcy occurs smoothly. As noted the process equilibrium production is given by $M^{equil} = S_C^{CM} = S_I^{IM}$, (12). As we approach the bankruptcy bifurcation the numerator in this M^{equil} expression, $S_I^{ext} D_O^{ext} -$

$S_O^{ext} D_I^{ext}$ goes smoothly to zero and the production drops to zero. In the VNM equilibrium model this zero production condition is included as (4). In our model this behaviour naturally emerges as a dynamical result when the process is unprofitable. We do not need to impose 'the dual cross adjustment' of the von Neumann model, the autonomous dynamical adjustment selects which processes remain and which "go bankrupt".

Furthermore at equilibrium the relation (15) means that both the input price and output price are equal to the total external demand divided by the total external supply. This non-trivial relationship implies that for example changing the external supply of the output S_O^{ext} will not only affect the output product price, but also the input product price. This is because the equilibrium processing rate, M^{equil}, is affected by changing S_O^{ext} which feeds back onto the input product price. This is of course providing the quantities are not changed so much that system goes bankrupt.

We have not yet mentioned the single process the funds $F(t)$ fixed points (14). These can be seen to be given by,

$$F^{CM,IM} = 2M^{equil} p_{min}^{CM,IM} \tag{18}$$

where $p_{min}^{CM,IM}$ denotes the fixed point price of the minimum product in the two phases. That is the funds fixed points $F^{CM,IM}$ are different in the two phases but are always given by the product of the fixed point of the minimum supply i.e. the equilibrium production M^{equil} which is the same in both CM and IM phases, and the fixed point price of the minimum product in the two phases, which is different in the two phases. In other words, the process equilibrium value is the product of the equilibrium price of the minimum supply and the equilibrium quantity of the minimum supply. Therefore when the minimum switches this can cause a crash in value.

4.3 Chain of Processes

We now come to consider what happens when multiple processes are coupled together. We show that in general we expect complex *multiple-timescale* dynamics to appear when multiple processes are coupled.

The simplest and most economically relevant system is the chain of processes in a fixed environment. Such a system might describe a factory production line. We show that this system has a novel multiple timescale limit cycle attractor and that accordingly we expect the output production to be dynamically rather complex.

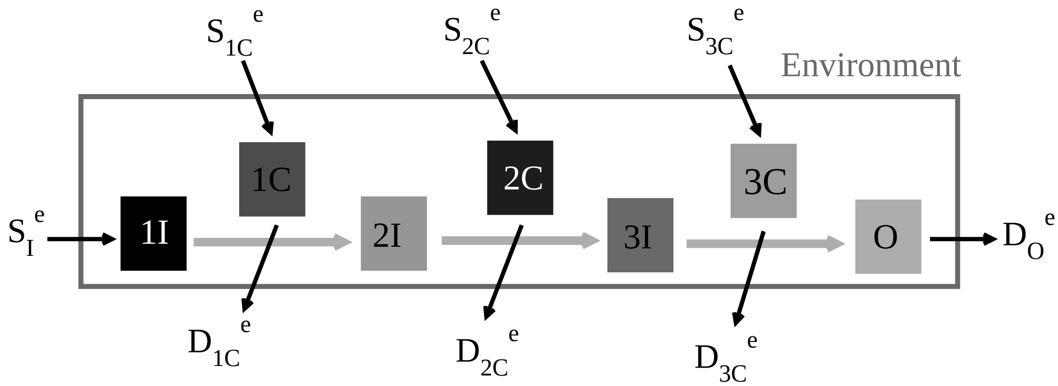

Fig. 2. Diagram of a 3 process chain with 7 products in fixed environment. 3 products are catalysts, one in each process each with a fixed external supply S_{iC}^{ext} and demand D_{iC}^{ext}, defining an external price for each catalyst. There is one input product I with a fixed external supply S_I^{ext} and one output product O with fixed external demand D_O^{ext}. There are two intermediate products with no external supplies and demands, but which are supplied by the feeding process and demanded by the drawing process.

Such a 3 process 7 product chain is shown in Fig.2, while the detail from the supplies time series for this system is shown in Fig.3. The time series grey scales refer to the same grey scales as in Fig.2. The time series shows a segment from a multiple timescale limit cycle attractor.

It is easy to understand how this behaviour is produced from the previous single process example. From Fig.1 we can see that as the slope of the line, ρ, containing the fixed points (12), increases, the amplitude and therefore period of the limit cycle increases. If the situation is such that each process has a different ρ parameter, then each process will have a different intrinsic oscillation frequency. We also know that the parameter ρ is given by (16) in terms of the equilibrium prices (15). For this 3 process system these quantities are given by,

$$p_{1C}^{equil} = \frac{D_{1C}^{ext}}{S_{1C}^{ext}}, \qquad p_{2C}^{equil} = \frac{D_{2C}^{ext}}{S_{2C}^{ext}}, \qquad p_{3C}^{equil} = \frac{D_{3C}^{ext}}{S_{3C}^{ext}}, \tag{19}$$

for each process catalyst prices. While the input and output fixed point prices for each process are,

$$p_I^{equil} = p_O^{equil} = \frac{D_O^{ext}}{S_I^{ext}}, \tag{20}$$

where we get this last relation, (20), from the fact that because of the input and output price equalization considerations above, processes in a chain must all have the same input and output equilibrium prices. Of course the system is not at equilibrium, but nevertheless the input and output prices in a chain should move approximately around the same long term average levels.

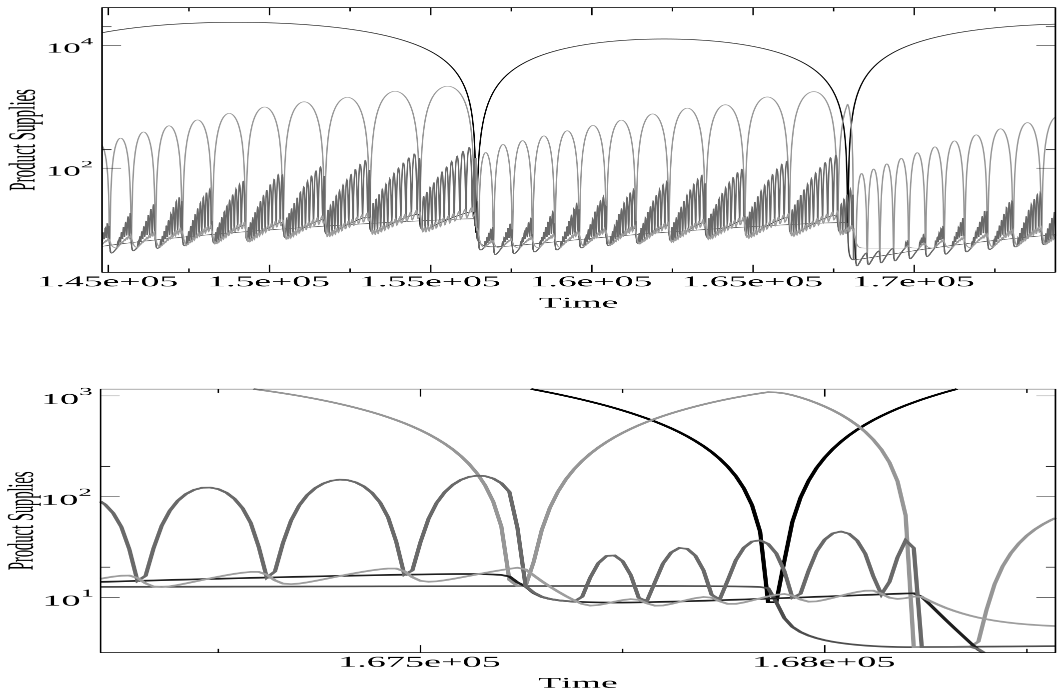

Fig. 3. The 6 time supplies time series for the 3 process chain described in the text and in Fig.2. The first process supplies, $S_{1I}(t)$ and $S_{1C}(t)$, move at lowest frequency, with $S_{1I}(t) > S_{1C}(t)$ most of the time. Similarly the second process supplies, $S_{2I}(t)$ and $S_{2C}(t)$, move at intermediate frequency with $S_{2I}(t) > S_{2C}(t)$ most of the time. Similarly $S_{3I}(t)$ and $S_{3C}(t)$ move at highest frequency, with $S_{3I}(t) > S_{3C}(t)$ most of the time. The lower panel is detail of top panel.

Now if we set the external environment such that for example,

$$p_{1C}^{equil} >> p_{2C}^{equil} >> p_{3C}^{equil} > p_I^{equil}, \tag{21}$$

then all processes oscillate with very different frequencies. The first process with the highest ρ value oscillates slowest the next at a higher frequency and so on. In this example the catalyst equilibrium price in the first process is 10 times the second which is 10 times the third, which is slightly greater than p_I^{equil}. These 3 different trapped frequencies are easy to see in Fig.3, where the first process switches on the slowest frequency, the second process at intermediate switching frequency and the third at the highest frequency.

In this case according to (12) since the external demand for the input D_I^{ext} is fixed to zero for convenience in all 3 of the processes (although the same dynamics appear when the intermediate products also have external demands and supplies) we get that the catalyst supply fixed point for process 1 is simply given by $S_{1C}^{CM} = S_I^{ext}$. In other words, the catalyst will oscillate around the external supply level as the process tries to keep the catalyst and the external supply level the same. Similarly for the second process catalyst fixed point we get, $S_{2C}^{CM}(t) = S_{1O}(t) = S_{1C}(t)$ (where now the 'fixed point' is slowly time

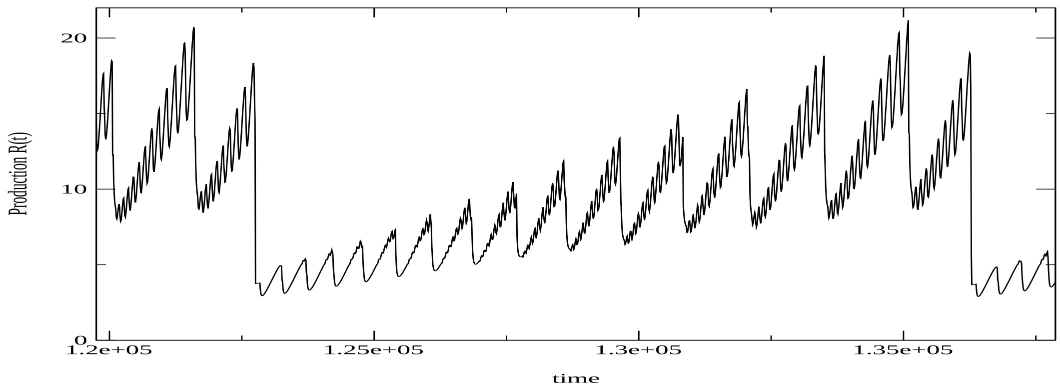

Fig. 4. Output production $M(t)$ time series from the 3 process chain described in the text.

varying) so that the second process catalyst supply will oscillate around the input supply which is the output supply from process 1 which is of course also the catalyst supply for process 1 since the catalyst is the minimum. Similarly we get $S_{3C}^{CM}(t) = S_{2O}(t) = S_{2C}(t)$ for process 3 and this explains why the catalyst levels all seem to get 'sewn' onto each other in Fig.3, in a hierarchical cascade.

Indeed that the catalysts should all oscillate around the same level in non-branching chains of processes is simply due to the fact that the fixed point of the minimum supply i.e. M^{equil}, the production rate, must be the same in each process in the chain. In this case where there is no external demand for input, so that $D_I^{ext} = 0$, $M^{equil} = S_I^{ext}$ for each process in the chain.

This dynamical complexity is therefore reflected in the network production $M(t)$ (10), which is the output supply from the last process in the chain shown in Fig.4. This is given by the output from the final process $M(t) = S_{3C}(t)$, which moves in a complex, apparently unpredictable way, around its fixed point, the input supply rate $M^{equil} = S_I^{ext} = 10$.

As mentioned, chains of supply processes with different price catalysts is a typical structure for a real economy, since most products are produced by a series of companies supplying parts to the next link in the chain. This example illustrates that even in the case of a fixed environment, as is the case here, with fixed input supplies and demands for all products the dynamics can be very complex and difficult to control. Control of such production processes has recently been addressed by Helbing [Helbing (2003)]. We expect such dynamics to therefore be typical of a real economy of coupled production processes.

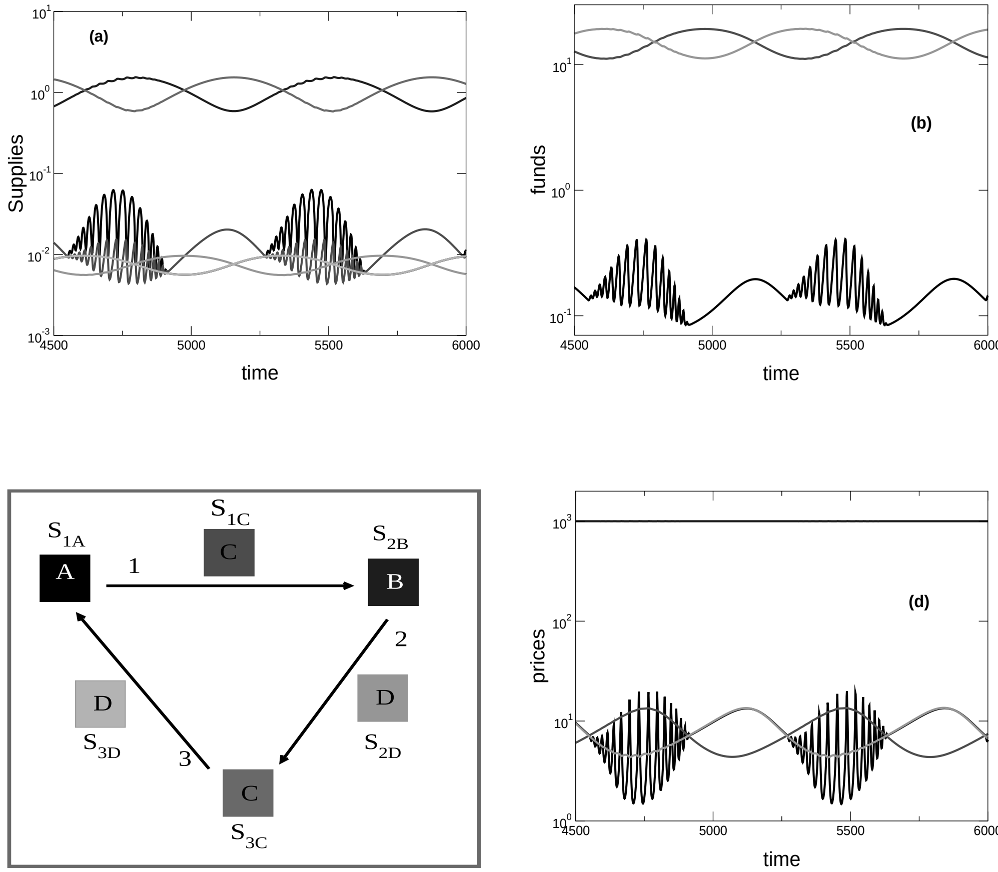

Fig. 5. Time series from process cycle described in bottom left panel (c). There are three processes denoted $1, 2, 3$ and four products A, B, C, D. Process 1 converts A into B via catalyst C. Process 2 converts B into C via catalyst D. Process 3 converts C into A via catalyst D. The squares in different greyscales correspond to the 6 individual process supplies, whose time series is shown in the top left panel (a). In that panel the high frequency oscillations occur in $S_{1A}(t)$ and $S_{1C}(t)$, which move around the same long term supply level as $S_{2D}(t)$ and $S_{3D}(t)$. The two supplies which move around much higher levels are $S_{2B}(t)$ and $S_{3C}(t)$. The top right panel (b) shows the 3 funds $F(t)$. The process 1, funds $F_1(t)$, shows the high frequency bifurcations. The process 2, $F_2(t)$, and process 3, $F_3(t)$, funds have a much higher level. The bottom right panel (d) shows the prices, $p_A(t)$ has high frequency oscillations and moves around the same level as , $p_B(t)$, and $p_C(t)$, but product D has much higher price level, $p_D(t)$.

4.4 Cycle of Processes

While economically unrealistic a simple cycle of 3 processes is useful for the illustration of the connection between multiple price levels and multiple timescales. In this example we will show that large differences in price levels between different products leads to the result that oscillations can have large amplitude in some processes but not be visible in others.

This system is, like the chain of processes, easy to understand simply by considering the fixed points. This process cycle is described in Fig.5(c). In this system product D is conserved by the cycle of 3 processes but may be demanded and supplied by the external environment, products A, B and C however are not supplied or demanded by the external environment. The total supply of $A + B + C$ is conserved by the system. Since A,B and C are in the same chain they must have same price fixed points. In this example we set the external supply S_D^{ext} and demand D_D^{ext} of product D such that the fixed point price of D, is large. We therefore expect the global price fixed points to be such that,

$$p_A^{global} = p_B^{global} = p_C^{global} << p_D^{global}. \tag{22}$$

where the ^{global} denotes global fixed points.

This is confirmed in Fig.5(d), where the 3 prices $p_A(t)$, $p_B(t)$ and $p_C(t)$ do move around the same general equilibrium level and $p_D(t)$ is seen to be much higher.

We also therefore get,

$$\rho_1^{global} = \frac{p_C^{global}}{p_A^{global}} << \rho_2^{global} = \frac{p_D^{global}}{p_B^{global}} = \rho_3^{global} = \frac{p_D^{global}}{p_C^{global}} \tag{23}$$

where the ρ_i^{global} is now the fixed point of the bifurcation parameter of the i process. In fact the actual bifurcation parameter $\rho_i(t)$ for each process as defined above, (16), is now time varying

$$\rho(t) = \frac{p_I^{equil}(t)}{p_C^{equil}(t)} = \frac{D_C^{ext}(t)(S_I^{ext}(t) + S_O^{ext}(t))}{S_C^{ext}(t)(D_I^{ext}(t) + D_O^{ext}(t))} \tag{24}$$

since the external environment of each process is time varying, and the external environment variables here in (24) are the appropriate ones for each process.

However we expect the bifurcation parameter $\rho_i(t)$ fixed points, ρ_i^{global} to obey (23) and we therefore expect process 1 to have much faster oscillation than processes 2 and 3. That this is so is apparent from Fig.5. In this example however, unlike the example of the previous 3 process chain, since the equilibrium prices $p_C^{equil}(t)$ and $p_A^{equil}(t)$ are time varying around the same level we expect the variation of $\rho_1(t) = p_C^{equil}(t)/p_A^{equil}(t)$ to cause a periodic bifurcation in process 1 as it periodically crosses unity. This is clear to see in the figure, where there is a periodic bifurcation from a high frequency oscillating state when $\rho_1(t)$ is slightly greater than 1 to a non-oscillating state where $\rho_1(t)$ is slightly less than 1.

Since the 3 processes are in the same chain M^{equil} the fixed point of the minimum supply, and the production rate, must be the same for each process. This is clearly seen for the supplies $S_{1A}(t)$, $S_{1C}(t)$, $S_{2D}(t)$ and $S_{3D}(t)$ in Fig.5(a) which move around the same level. The catalysts $S_{2D}(t)$ and $S_{3D}(t)$ are always the minimum supplies in processes 2 and 3 respectively, while the minimum supply of process 1 varies between the input supply $S_{1A}(t)$ and the catalyst supply $S_{1C}(t)$ as can be seen.

Furthermore according to (18) the funds $F_i(t)$ fixed points are given by the product of the production rate and the price of the minimum rate controlling product. Again according to (22) since the price of D is much greater than the prices of both A and C the funds fixed point of process 1 will be much less than 2 and 3, $F_1^{IM,CM} = 2M^{equil}p_{A,C}^{global} << F_{2,3}^{CM} = 2M^{equil}p_D^{global}$. Again this relationship is clearly seen in Fig.5(b).

It is this large disparity in funds $F(t)$ between the 3 processes which explains the fact that oscillations are seen in some processes but not others. In particular for example we see from the prices time series that $p_A(t)$ shows the high frequency oscillations from process 1 while $p_B(t)$ and $p_C(t)$ do not, although all three product prices are affected by process 1. In fact for $p_A(t)$ we have,

$$p_A(t) = \frac{S_{1A}(t) + Min(S_{3C}(t), S_{3D}(t))}{1/2F_1(t)} = \frac{S_{1A}(t) + S_{3D}(t)}{1/2F_1(t)},$$

since $S_{1A}(t)$ and $S_{3D}(t)$ are of the same order of size, the fast oscillations in $S_{1A}(t)$ and $F_1(t)$ show up in $p_A(t)$. However for $p_C(t)$

$$\begin{aligned} p_C(t) &= \frac{S_{1C}(t) + Min(S_{2B}(t), S_{2D}(t)) + S_{3C}(t) - Min(S_{3C}(t), S_{3D}(t))}{1/2F_1(t) + 1/2F_3(t)} \\ &= \frac{S_{1C}(t) + S_{2D}(t) + S_{3C}(t) - S_{3D}(t)}{1/2F_1(t) + 1/2F_3(t)} \sim \frac{S_{3C}(t)}{1/2F_3(t)}, \end{aligned}$$

so that the high frequency oscillations of the process 1 variables are “swamped” by the much larger magnitude process 3 variables which do not have high the frequency component, and $p_C(t)$ does not show the high frequency oscillation.

This combination of bifurcations between different frequency states and the filtering of oscillations by threshold type prices, when some processes have much larger magnitudes than others, means time series from random economic networks show novel multiple timescale complex switching dynamics.

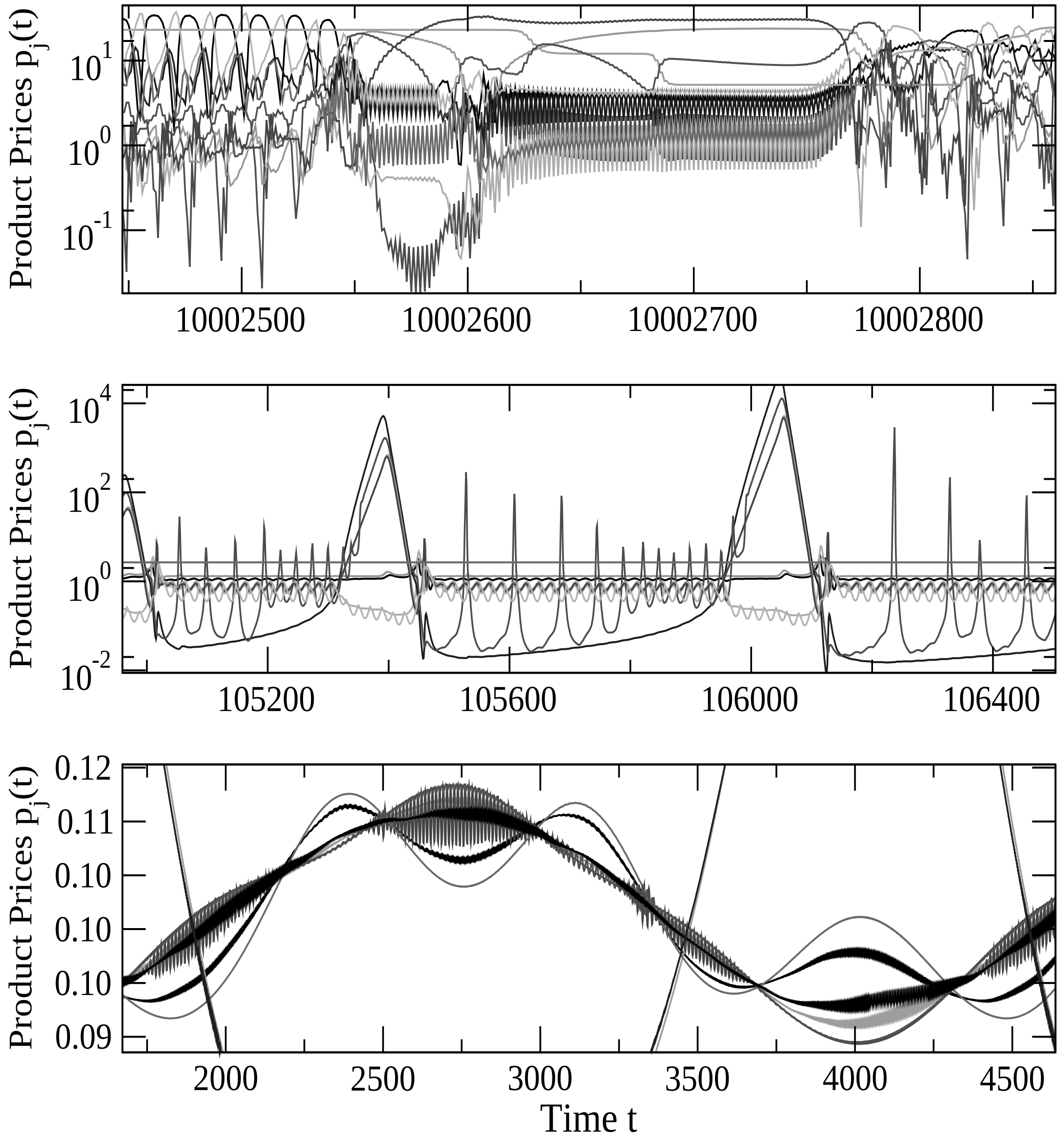

Fig. 6. Examples of the time series of prices for a network model, consisting of 15 processes and 10 products. The connection matrix is generated as in the text, and three examples of time series of such networks are plotted.

4.5 Dynamics of Process Networks

Real economic production networks are of course more complicated than the cyclic process system discussed in the previous subsection. As mentioned in §2, a production process is a transformation of some commodities into other commodities via intermediate processes. The connections between commodities and processes are given by the input and output matrices $A = a_{ij}$ and $B = b_{ij}$ respectively. One can consider various examples of such networks, and study their dynamic behaviour. Indeed, by assigning connections between processes and products randomly, and constructing the corresponding matrices

A and B, we can make an example of such a network. The only requirement are that each product should be an input to at least one process and an output from at least one process, and that each process should have at least one input product and one output product. We have numerically studied many such examples. The behaviour, of course, depends on the particular network. For most networks, however, we have observed complex oscillatory dynamics as well as successive switching behavior from one type of oscillation to others. Such behaviour is quite common in such networks.

Although it is not so easy to make a detailed analysis of the dynamics of such networks of processes here, it will be useful to give a few typical example of such dynamic behavior, showing the time series of the prices for three different networks consisting of 15 processes and 10 products. As shown in Fig.6, the time series shows switching over several states, where oscillations on multiple timescales are involved. We find some processes controlling bifurcations in other processes in a hierarchical way. This type of hierarchial structure with multiple time-scales is commonly observed in the network model. Note again that even in this network model the dynamics does not show price divergence and economic production is dynamically sustained. The resultant dynamics, as seen in Fig.6, however, is extremely complex and the detailed analysis will be addressed in future work.

We note that a model with 15 processes and 10 products is too small for a real economic model. Still, it is important to note that even this model, with such a small network, can generally show very complex dynamics with hierarchial multiple time-scales. A more complicated network model will include at least this type of dynamics. Hence this result is expected to provide a basis for considering a hierarcy of time-scales in economic processes in general.

5 Discussion

In the present paper we have explicitly constructed a dynamical model based on the idea of von Neumann for an economic production network.

His original model only discussed the equilibrium state depending on the free goods rule and the profitability rule. We discarded these rules in order to discuss the dis-equilibrium state, and we replaced them with a simple procedure for each production process to allocate its assets and a simple pricing rule. Even with these simple procedures we found the system shows reasonable behaviour. For example we found that, if in the absence of the process the external environment is such that $p_I^{brupt} < p_O^{brupt}$, then providing the process is technologically feasible it will appear and grow in production until $p_I^{equil} = p_O^{equil}$.

It is worth while noting that our model has no rational agents who optimize their utility. Even so, we observe the above reasonable economic phenomena. These results remind us of the conclusions of Padgett *et al.* (2003), that a complex production network can emerge even if the agents have very simple decision making process.

In this model we have found interesting oscillatory dynamics. One origin of these complex dynamics lies in the "minimum" term in (10). This is in contrast with the rate equation in standard chemistry, where the rate of a process is given by the Law of Mass Action, i.e., the reaction rate increases when the concentration of any of the reagents increases.

Still, this "minimum" condition need not necessarily be so strict. We can generalise this minimum condition (10) to a somewhat smooth function such as

$$M_i(t) = M(\{x_j\}) = (\sum_{j=1}^{N} x_j^{-r})^{-1/r}$$

which becomes the minimum (10) in the limit $r \to \inf$. Indeed the results we described here are reproducible as long r is large enough. In economic terms this means that the strict L-shape isoquant of production function is not necessary for the complex behaviour of our model. Smoother (neoclassical) production functions can also produce the complex oscillation observed here.

We must point out that this production function characterisitic doesn't only cause the observed complex behaviour, but also insures the absence of divergence. Variations of economic production, or prices with large amplitude appear, but at the same time the total destruction of economic production is avoidable. This important macro-level dynamic stability is insured by the shape of production function.

As we mentioned earlier, in our model the way the processes allocate their funds between the different products they require is not directly affected by prices. This assumption seems too strong for an economic model. However, this assumption is not essential to the overall behaviour of our model. For example, instead of using $F_i(t)a_{ij}$ as the process i demand for product j, we introduce the term $F_i(t)\sigma_{ij}(t)$ such that,

$$\sigma_{ij}(t) = \gamma \frac{p_j(t)a_{ij}}{\sum_j p_j(t)a_{ij}} + (1-\gamma)a_{ij}$$

to take into account the price feedback. (The model we studied in the paper corresponds to the case with $\gamma = 0$). Indeed we have also studied the above model with $\gamma \neq 0$. Still the complex cyclic behaviour we presented here is preserved with this extension, and the results here are qualitatively unchanged. We suspect that no matter what prescription the processes use to allocate their demand ratios $\sigma_{ij}(t)$, the multiple timescale complexity we observed will be a generic factor of production systems and production networks.

The complex oscillation itself is interesting as a dynamical system. The oscillation is sustained without divergence by switching over a succession of temporal evolutions defined by each segment in the phase space, that is, by the specific choice of the minimum condition. Such oscillations created by the switching mechanism will be rather universal. Also, in high dimensional dynamics where there are many degrees of freedom, the minimum condition produces a complex segmentation in the phase space, which may lead to complex oscillatory dynamics with multiple time scales, as observed here. Elucidation of this complex oscillation will be an important problem in nonlinear dynamics theory.

Our model demonstrated the fact that the switching mechanism in the production function results in the multiple time scale fluctuation at higher levels. This view can open a new pathway to understand economic dynamics in real economic history. Fernand Braudel, a great French socio-economic historian, proposed the idea that there are layers of 'duration' in the history of capitalism [Braudel (1979)]. Our model can supply a mathematical background for his view.

6 Acknowledgements

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