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Functional dynamics
II: Syntactic structure

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Abstract

Functional dynamics, introduced in a previous paper [Physica D 138 (2000) 225–250] is analyzed, focusing on the formation of a hierarchical rule to determine the dynamics of the functional value. To study the periodic (or non-fixed) solution, the functional dynamics is separated into fixed and non-fixed parts. It is shown that the fixed parts generate a one-dimensional map by which the dynamics of the functional values of some other parts are determined. Piecewise linear maps with multiple branches are generally created, while an arbitrary one-dimensional map can be embedded into this functional dynamics if the initial function coincides with the identity function over a finite interval. Next, the dynamics determined by the one-dimensional map can again generate a 'meta-map', which determines the dynamics of the generated map. This hierarchy of meta-rules can continue recursively. It is also shown that the dynamics can produce 'meta-chaos' with an orbital instability that is stronger than exponential. The relevance of the generated hierarchy to biological and language systems is discussed, in relation with the formation of syntax of a network.

Keywords: Functional dynamics; Syntactic structure; Iterated map; Self-reference

1. Introduction

In a previous paper [1] (to be referred as I), we introduced functional dynamics to investigate the articulation process carried out on an initially inarticulate network. In the functional dynamics, objects and rules are not separated in the beginning, and we study how objects and rules appear from an inarticulate network through iterations of functions. The functional dynamics provides a simple universal model for this appearance.

In the general introduction of the paper I, we discuss five requisites for a biological system, while two of them are explicitly studied. These are the following:

- Inseparability of rule and variable ($f_n \circ f_n$).
- The articulation process from a continuous world.

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In I, we investigate functional dynamics defined by

$$f_{n+1} = (1-\epsilon) f_n + \epsilon f_n \circ f_n. \tag{1}$$

The evolution of the function f_n has been studied with n representing the iteration step and ϵ as a parameter. We have shown that an articulation process is generated in this one-dimensional functional dynamics. As f_n evolves, first, type-I fixed points satisfying $f_{n+1}(x^{\mathrm{I}}) = f_n(x^{\mathrm{I}}) = x^{\mathrm{I}}$ are formed, and then type-II fixed points satisfying $f_{n+1}(x^{\mathrm{II}}) = f_n(x^{\mathrm{II}}) = x^{\mathrm{I}}$ are formed [1]. The articulation process is studied as a classification process of how f_n converges to constant on distinct intervals consisting entirely of type-II fixed points. For a given value $a = f_n(x)$, the inverse set $I_n = f_n^{-1}(a)$ is given as an articulated class. This means that the filter articulates the continuous world x into some segments according to the value $f_n(x)$. For such sets, $I_{n+1} \supseteq I_n$ holds, and the dynamics of this system is determined completely by a set of relations among these intervals as $n \to \infty$. This reduction of the degrees of freedom out of a continuous world is the articulation process. This articulation is most clearly seen in the relation between type-I and type-II fixed points. Intervals of type-II points corresponding to the same type-I fixed points are generated from an initial continuous function.

As an articulation process, a structure independent of n is formed by the fixed points, while the functional values of some points change periodically in time, taking the values of different type-II fixed points successively (i.e., being mapped to the rigid structure constituted by fixed points). This periodic function provides an example of how objects and rules depend on each other, based on a rigid structure unchanged under iteration.

In the present paper, we focus on a dynamical aspect of these functional dynamics, to study how rules for dynamic change emerge through iteration of the functional mapping. With this approach, we investigate the third and forth requisites mentioned in the general introduction of I:

- Formation of a rule to change the relation among objects.
- Formation of hierarchical rules.

From our viewpoint, objects and rules emerge from the same level in which code and encoding circulate dynamically. For example, in natural language, there is a set of words and rules that forms sentences. When we assume in the beginning that the objects and rules are already separated, the theory of language has to be based on formal languages [4,5], since the structure of the language has to be studied without resorting to the objects. With the separation of words and rules, one neglects the fact that the rules have to be described by using words, while the meaning of a word has to be described by a sentence. This implies that a theory starting from a hierarchy in which objects and rules are separated is not sufficient as a mathematical framework for natural language. Natural language is described as an assembly of objects, rules, meta-rules, meta–meta-rules, and so forth, while this hierarchy is not given in advance. Indeed, this process of emergence of hierarchical structure from pre-structured “something” is a common characteristic of biological systems, as is seen, for example, in the hierarchy of cell, tissue, organism, and so forth, starting from an assembly of chemical reactions. A mathematical formulation is required to study the hierarchy of the successive formation of rules at successively higher levels [6,7].

By taking the same viewpoint as that in I, we study a network of input–output relationships (for example in the language/objects) as a functional form. With the evolution of the functional dynamics, some structure is constructed step by step. In particular, we study how a hierarchy of rules and meta-rules emerges through the iteration. Here, a structure is formed first by the configuration of fixed points, while a rule for the dynamic change of the structure is organized according to an ‘orbit’ of the functional dynamical system, and then a meta-rule is formed governing the dynamic structure generated by the orbit.

This implies that once we get a rigid (fixed-point) structure, unchanged under repeated mapping, as the elementary part of the dynamical network, hierarchical structure appears under some restrictions. Here, the rigid structure and hierarchical structure correspond to the articulated objects and the operations acting on the objects, respectively. This separation of objects and rules emerges, since we extract the rigid structure out of the functional dynamics.

The present paper is organized as follows. In Section 2, we explain the basic properties of the functional dynamics again to facilitate its representation by the introduction of a ‘generated map’. With the generated map, it is shown in Section 3.1 that a one-dimensional map is embedded in the functional dynamics. This map works as a rule governing the change of the functional values over some intervals. In Sections 3.1 and 3.2, piecewise linear maps called the Nagumo–Sato map and the ‘multi-branch Nagumo–Sato map’ are naturally embedded in the functional dynamics. In Section 3.3 a larger class of one-dimensional maps is embedded into our functional map. This implies that chaotic functional dynamics is possible in our system. By choosing a suitable initial condition, it is shown in Section 4 that the functional dynamics can form a hierarchical structure. A meta-rule for the change of the functional values is formed which changes according to the (chaotic) dynamics generated by the one-dimensional map. The maps can be nested recursively and generate higher level meta-maps successively. In Section 5, we discuss syntactic structure derived from this functional dynamics, and the relevance of our results to the third and fourth requisites of biological systems mentioned above.

2. Model

The functional map Eq. (1) has the form

$$f_{n+1} = F(f_n, f_n \circ f_n). \tag{2}$$

Here we study some characteristics of this functional equation with a one-dimensional f_n. In connection with our motivation for biological systems and language structure, the function f_n is considered to represent an abstraction of the input–output relation network, while $f_n \circ f_n$ provides a self-referential term. Since we are interested in modeling the situation in which code and encoding are not separated, f_n represents a projection from a set into itself.

First, we discuss two characteristic properties of this equation:

- If images of x' and x'' by the function f_n have the same value $f_n(x') = f_n(x'')$ at n, the subsequent evolutions of $f_m(x')$ and $f_m(x'')$ (with $m > n$) are identical, because dynamics are determined completely by the value $f_n(x)$.
- Eq. (2) can be split as

$$f_{n+1}(x) = g_n(f_n(x)), \qquad g_n(x) = F(x, f_n(x)). \tag{3}$$

The first property above implies the ability of articulation of this system. Once f_n identifies x' and x'' as the same thing, the two points evolve in the same way. The second property provides a novel viewpoint to study this model. With this separation, one can say that a point $f_n(x')$ evolves under g_n, which is determined from f_n itself. This is a characteristic property of this functional equation. Given f_n, a map g_n is determined. The function f_n evolves to f_{n+1} under the map and g_{n+1} is determined. In this paper, the term ‘function’ is used to represent f_n, while a ‘generated map’ is used in reference to g_n. The “value of x'” indicates $f_n(x')$ in the present paper.

If $f_n(x')$ is a fixed point for n at x' (which does not mean a fixed function as a whole), the generated map of $f_n(x')$ is a fixed generated map at the point x'. To study the functional dynamics, we first study a fixed generated map $g_n(x')$ and see how other points x'' evolve under this generated map (Section 3). Second, we study the case in which $g_n(x')$ itself changes in time, by taking a suitable initial configuration of f_0. There, a hierarchical structure (meta-map) is considered (Section 4).

For simplicity, we impose one more restriction on (2) following I. We assume that if the relation $f_n(x')$ is fixed, it satisfies the relation $f_n(x') = f_n \circ f_n(x')$. This condition implies that the change of $f_n(x')$ vanishes when the self-reference of a function agrees with the function itself. One of the simplest models of this type is (1), obtained by choosing the form

$$F(x, y) = (1 - \epsilon)x + \epsilon y \tag{4}$$

with $0 < \epsilon < 1$. (The case with a general form for $F(x, y)$ is briefly discussed in Appendix A and will be discussed in a future paper.)

For the type of model we study, the dynamics relax toward the self-consistent relation $f(x') = f \circ f(x')$. For the remainder part of this paper, we focus on the functional dynamics (1). In this case, the generated map is given by $g_n(x) = (1 - \epsilon)x + \epsilon f_n(x)$.

Now we obtain two useful properties:

- A value $f_n(x')$ which satisfies the condition $f_n(x') = f_n \circ f_n(x')$ is a fixed point for n, i.e., $f_{n+1}(x') = f_n(x') \equiv f(x')$. Here we denote fixed function by f instead of f_n, and fixed point for n by $f(x')$ without the suffix n.
- There is a transformation T ($\mathbf{R} \to \mathbf{R}$) which satisfies the condition $F(T(x), T(y)) = T \circ F(x, y)$. (The explicit form of T is discussed below.)

Here, all the points x^{I} which satisfy $f(x^{\mathrm{I}}) = x^{\mathrm{I}}$ are fixed points for n of the functional equation (1). Since all points x with an identical value $f(x)$ evolve identically, all the points x^{II} that satisfy $f(x^{\mathrm{II}}) = x^{\mathrm{I}}$ are again fixed points for n. For convenience, we have classified (see I) these fixed points as follows:

- x^{I} is a type-I fixed point with $f(x^{\mathrm{I}}) = x^{\mathrm{I}}$. We denote the set $\{x^{\mathrm{I}}\}$ by I^1.
- x^{II} is a type-II fixed point with $f(x^{\mathrm{II}}) \in I^1$, and satisfying $x^{\mathrm{II}} \notin I^1$. We denote the set $\{x^{\mathrm{II}}\}$ by I^2.

A type-I fixed point is a point at which the graph of f_n intersects the identity function. This 'type' is extended to arbitrary type-N. We define a type-N point as a point which satisfies the condition $f_n(x^N) \in I^{N-1}, x^N \in I \backslash \cup_{i=1}^{N-1} I^i$, after the transient in the functional dynamics has died away. Here x^N represents a type-N point and I^N represents a set of type-N points $\{x^N\}$ (type-N interval). [1] Although type-I and type-II points are fixed points, type-N ($N > 2$) points cannot be fixed points. In fact, if x is a fixed point and $y = f(x)$, the fixed point condition is written $y = f(y)$, which means y is a type-I fixed point and x is a type-I or type-II fixed point. For convenience, we call a partial function defined on a set of type-N points ($f_n|_{I^N} \equiv \{f_n(x)|x \in I^N\}$) a type-$N$ function.

In I, we introduced the concept of a 'self-contained section' (SCS), which is defined as a connected interval I such that $f_n(I) \subset I$, while no connected interval $J \subset I$ satisfies $f_n(J) \subset J$, and $f_n(I + \delta) \subset I + \delta$ for arbitrary small δ, either. Here, we extend this definition to introduce the 'closed section' (CS) and 'closed generated map' (CGM). A CS is defined as a set I such that $f_n(I) \subset I$ (where I is not necessarily connected), while a CGM is defined as a part of a generated map $g_n|_J$ such that $g_n(J) \subset J$ (where J is not necessarily connected). In Eq. (1), if $f_0(I) \subset I$, then $f_n(x) \in I$ for all n, and I is a CS. If a CS I is connected, the partial generated map $g_n|_I$ becomes CGM.

Now let us return to the transformation T. From Eq. (4), it is straightforward that $F(T(x), T(y)) = T \circ F(x, y)$ is satisfied by choosing the linear transformation

$$T(x) = ax + b. \tag{5}$$

In fact, Eq. (1) assumes the form of an operation of taking a weighted average of $f_n(x')$ and $f_n \circ f_n(x')$. Thus, the functional dynamics are invariant under a scaling transformation in which x and $y = f(x)$ are multiplied by the same factor and shifted by the same value.

[1] Note that a type of a point is defined at each time step n.

The above invariance $F(T(x), T(y)) = T \circ F(x, y)$ means that T and F commute. Under this transformation, a connected CS (x_1, x_2) is shifted to $(ax_1 + b, ax_2 + b)$ giving a new CS. The above invariance will be used to embed a one-dimensional map into this functional dynamics in Section 3 and to construct a meta-map in Section 4.

3. One-dimensional map in functional dynamics

3.1. General properties

In I, we found that the function f_n does not converge to a fixed function as $n \to \infty$ for some initial functions. For example, for the initial function $f_0(x) = rx(1-x)$, f_n does not converge to a fixed function for some range of parameter r referred to as the R (random) phase in I, where the number of discontinuous points of f_n and the length of f_n increases in proportion to M, the number of mesh points adopted for the numerical calculation.

Recalling that the f_n for large n looks almost random in the R phase, we have also computed f_n from Eq. (1) using random initial conditions, as an extreme case. For such initial conditions, we divide the interval $[0, 1]$ into M intervals as $([i/(M-1), (i+1)/(M-1)])$ and choose the value of $f_n(i) \in [0, 1]$ randomly. An example of $f_\infty(x)$ for an interval x and the return map $(f_n(x), f_{n+1}(x))$ for the interval are displayed in Fig. 1. Here, $f_n(x)$ mainly consists of many flat intervals with the same value, while for some points x', $f_n(x')$ changes periodically in time. As plotted by the return map, it is found that the periodic dynamics obey a certain rule. As shown in the inset of Fig. 1(b), a clear piecewise linear structure is visible in the return map. In this section, we study how this type of return maps is generated (Section 3.2) and investigate the class of maps that can appear with these functional dynamics (Section 3.3).

A fixed generated map can be constructed from type-I and type-II fixed points. This fixed generated map acts as a one-dimensional map $g_n|_{I^1 \cup I^2}$. To extract the temporal change of other x points, it is useful to think of the interval I as the union of three parts: $I = I_{\rm map} \cup I_{\rm driven} \cup I_{\rm rest}$.[2] Here, $I_{\rm map}$ is a set such that $f_n|_{I_{\rm map}}$ is a fixed function, $I_{\rm driven}$ is a set such that $f|_{I_{\rm driven}} \subset I_{\rm map}$ and $I_{\rm rest}$ is the rest: $I_{\rm rest} = I \setminus \{I_{\rm map} \cup I_{\rm driven}\}$. The fixed function $f|_{I_{\rm map}}$ determines a fixed generated map $g|_{I_{\rm map}}$ which determines the time evolution $f_n|_{I_{\rm driven}} \to f_{n+1}|_{I_{\rm driven}} (f_{n+1}|_{I_{\rm driven}} = g(f_n|_{I_{\rm driven}}))$.

First, we assume there exists only one type-I fixed point $x^{\rm I}(f(x^{\rm I}) = x^{\rm I})$ and one type-II fixed point $x^{\rm II}$ and that $f(x^{\rm II}) = x^{\rm I}$. The generated map at $x = x^{\rm II}$ is $g(x^{\rm II}) = (1-\epsilon)x^{\rm II} + \epsilon x^{\rm I}$, which can be rewritten as $g(x) = (1-\epsilon)(x - x^{\rm I}) + x^{\rm I}$ by inserting $x = x^{\rm II}$.

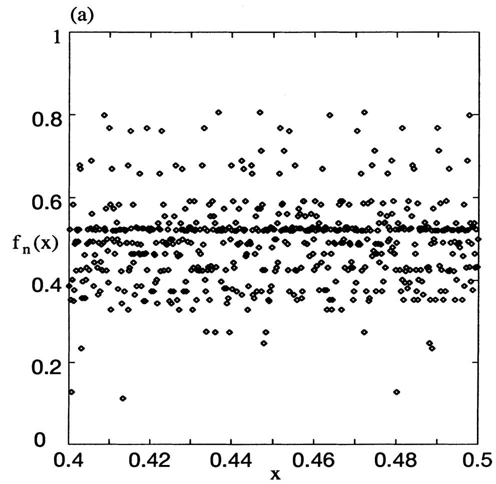
Next, we assume that there is an interval $I_{\rm map}$ consisting entirely of a single type-I fixed point and corresponding type-II fixed points, which satisfy $f(x'') = x^{\rm I}$ for $x'' \in I_{\rm map}$. Assuming the existence of such an interval, the generated map is given by $g(x) = (1-\epsilon)(x - x^{\rm I}) + x^{\rm I}$ for this interval ($x \in I_{\rm map}$). The graph of $g|_{I_{\rm map}}$ is a line with a slope $1-\epsilon$ that intersects the type-I fixed point. This line that is used as generated map is determined by the configuration of type-II fixed points. If the interval $I_{\rm map}$ is connected for $x \in I_{\rm driven}$ all $f_n(x)$ evolve to $f_\infty(x) = x^{\rm I}$.

Now consider the more general case in which an interval $I_{\rm map}$ consisting of several type-I fixed points and several subintervals of type-II points that are mapped to one of the type-I fixed points. In this case, the generated map is determined by the arrangement of type-I and type-II fixed points. This map is a piecewise linear function with slope $1-\epsilon$, which intersects the type-I fixed points (see Fig. 2). Here, we consider the following two cases for the configuration of type-I fixed points:

- There exist a finite[3] number of type-I fixed points (Section 3.2).
- There exist a finite number of type-I fixed intervals (Section 3.3).

[2] As will be shown, the evolution of $f_n(x')$ for $x' \in I_{\rm driven}$ is determined by $\{g(x)\}$ with $x \in I_{\rm map}$. In this sense, we call this interval 'driven' by the interval $I_{\rm map}$.

[3] It is not difficult to extend the following argument to the case with a countable number of type-I fixed points.



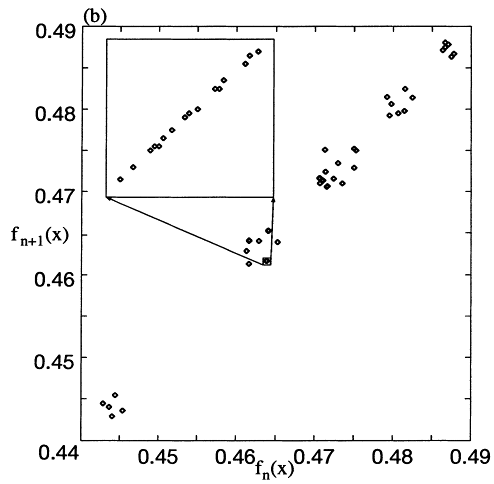


Fig. 1. (a) The graph of the $f_{50\,000}(x)$ $(0.4 < x < 0.5)$ for random initial f_0 with $M = 6000$ and $\epsilon = 0.02$. It consists of type-I fixed points, type-II fixed intervals and some periodic points. (b) A part of the return map of (a) for all $(f_n(x), f_{n+1}(x))$ $(n = 50\,000$–$50\,050)$. The return map consists of some points and lines that have slope $(1 - \epsilon)$ (in the inset).

A type-I fixed interval is a connected interval $[a, b]$ on which $f(x) = x$ for all $x \in [a, b]$. A type-I fixed point is the limiting case of a type-I fixed interval (i.e., in which $a = b$).

In the first case, let us denote elements of I^1 by x_i, where $i < j$ implies $x_i < x_j$. The type-I fixed intervals are ordered in the same way as the fixed points, as $I_0^1, I_1^1, \ldots, I_{m-1}^1$, where $i < j$ implies $\max I_i^1 < \min I_j^1$.

Depending on the configuration of type-I and type-II fixed points, a one-dimensional map is determined. If $g|_{I_{\mathrm{map}}}$ is a CGM, $f|_{I_{\mathrm{driven}}}$ evolves under the one-dimensional map. In the next section, we study the case with isolated type-I fixed points, while in Section 3.3 we discuss the case with type-I fixed intervals.

3.2. Case with finite type-I fixed points

In this section, we consider the case with finite type-I fixed points. As shown in I, the function f_n often tends to approach a piecewise constant function, consisting of a discrete set of type-I fixed points and several intervals

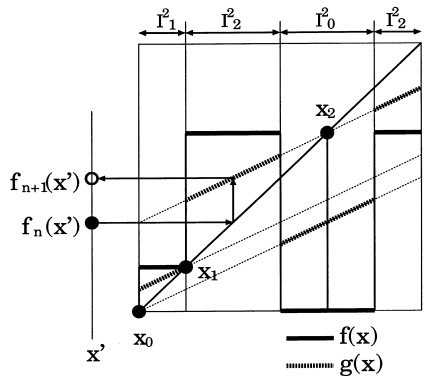


Fig. 2. An example of three type-I fixed points ($x_0 < x_1 < x_2$) and $f(x)$ for $x \in I$. The map $g(x)$ is represented by the dotted line. $g(x)' = (1-\epsilon)$ and is constant. Dynamics of another point $f_n(x') \in I$ is determined by $g(x)$.

of type-II fixed points at which $f(x)$ assumes the same value. Thus the existence of such type-I fixed points and type-II intervals is common in our model ([1] and Fig. 1(a)).

Corresponding to type-I fixed points x_i, we define sets of type-II fixed points I_i^2 to be those satisfying $f(I_i^2) = x_i$, where I_i^2 is not necessarily connected and consists of several connected intervals in general. Since f_n is a single-valued function, there is no intersection among I_i^2 and I^1. Now, the interval I is the union of I^1, I^2, I_{driven} and $I_{\mathrm{rest}}(I = I^1 \cup I_0^2 \cup I_1^2 \cup \cdots \cup I_{m-1}^2 \cup I_{\mathrm{driven}} \cup I_{\mathrm{rest}})$. Following the argument in the last section, the generated map on the interval $I_{\mathrm{map}} = I^1 \cup I^2$ has the form

$$g(x) = \begin{cases} g[0](x) & = (1-\epsilon)(x-x_0)+x_0 & \text{for } x \in I_0^2, & \text{where } f(I_0^2) = x_0, \\ g[1](x) & = (1-\epsilon)(x-x_1)+x_1 & \text{for } x \in I_1^2, & \text{where } f(I_1^2) = x_1, \\ \vdots & \vdots & \vdots & \vdots \\ g[m-1](x) & = (1-\epsilon)(x-x_{m-1})+x_{m-1} & \text{for } x \in I_{m-1}^2, & \text{where } f(I_{m-1}^2) = x_{m-1}, \\ g(x) & = x_i & \text{for } x \in I^1, & \end{cases} \tag{6}$$

where $[i]$ denotes a line corresponding to the type-I fixed point x_i. Each $g[i]$ is referred to as an 'i-branch'.

As discussed above, this generated map acts as the evolution rule for points x' that are mapped to one of the type-II fixed points [i.e., $f_n(x') \in I_i^2$, or, in other words, $f_{n+1}(x') = g(f_n(x')) = (1-\epsilon)f_n(x') + \epsilon x_i$ (see Fig. 2)].

The combination of some type-I fixed points and an set of type-II fixed intervals satisfying certain conditions can give a CGM. Here we assume there exist m type-I fixed points ($x_0 < x_1 < \cdots < x_{m-1}$), and the points in the interval (x_0, x_{m-1}) are assumed to be mapped to one of the type-I fixed points. Then, according to (6), for $i = 0, 1, \ldots, m-1$, $g[i]|_{I_i^2} \subset [x_0, x_{m-1}]$, because of $x_0 \le (1-\epsilon)x + \epsilon x_i \le x_{m-1}(x \in (x_0, x_{m-1}))$. As a total, $g|_{I^1 \cup I^2} \subset I^1 \cup I^2$, and the $g|_{I^1 \cup I^2}$ becomes a CGM. Thus, for $x' \in I_{\mathrm{driven}}$, the evolution of $f_n(x') \in I_i^2$ is determined by the CGM. This $f_n(x')$ is included in a type-II fixed interval, and thus x' can be called type-III. The pre-image $f_n^{-1}(I_i^2)$ is denoted as $I^3 (= I_{\mathrm{driven}})$. Now, the interval I can be written as $I^1 \cup I^2 \cup I^3 \cup I_{\mathrm{rest}}$. Here we call this type of configuration of $\{f(x)\}$ that determines a closed one-dimensional generated map as 'unit-I'. The situation is drawn schematically in Fig. 3.

For example, we assume that there are two type-I fixed points, x_0 and $x_1 (I^1 = \{x_0, x_1\})$. We divide the interval $I = [x_0, x_1]$ into I^1, I_0^2, I_1^2, I^3 and I_{rest}. Since $f(x)$ has a value x_0 or x_1, in I^2, $g|_{I^1 \cup I^2}$ has a '0-branch' and a '1-branch'. The map $g|_{I^1 \cup I^2}$ has the same slope $(1-\epsilon)$ on $I^1 \cup I^2$. This class of map includes the Nagumo–Sato map [8].

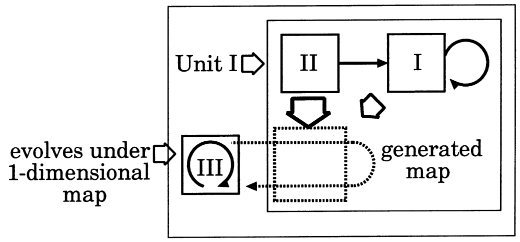


Fig. 3. The schema of the generated map. The configuration of type-I and type-II fixed points determine the map, while the evolution of the type-III points are determined by the generated map.

The Nagumo–Sato map is given by the equation

$$x_{n+1} = kx_n + w \pmod 1 \tag{7}$$

with $0 < k < 1$ and $0 < w < 1$. This map has two branches with the same slope k for the interval $[0, (1-w)/k]$ (the first branch) and $[(1-w)/k, 1]$ (the second branch) (see Fig. 4). To have these two branches, we need two intervals of type-II fixed points. With the aid of transformation (5), the domain of f_n is restricted to [0, 1], where two type-I fixed points are located at 0 and 1, without loss of generality. Our purpose here is to show that the generated map at the type-II fixed intervals on I_0^2 and I_1^2 has the form of Eq. (7).

With the transformation (5), the slope $k = 1 - \epsilon$ is conserved. Transforming (7) by multiplying by ϵ and shifting by $1 - w$ along x and $y = f_n(x)$-directions, we can embed the Nagumo–Sato map (of the interval size ϵ) into $g(x)$. The required condition is $I_0^2 = [(1-w)/(1-\epsilon), 1 + \epsilon - w]$ and $I_1^2 = [1 - w, (1-w)/(1-\epsilon)]$ (see Fig. 4). This map $g|_{[1-w, 1-w+\epsilon]}$ becomes a CGM. Note that this situation can generally arise without choosing a very special initial function. This is why the functional dynamics from arbitrary initial conditions often lead to a periodic cycle governed by the Nagumo–Sato map as in Fig. 1.

An example of our simulation is displayed for $\epsilon = 0.2$ and $w = 0.44$ in Fig. 5, where the discontinuous point ($a = (1-w)/(1-\epsilon)$) of the Nagumo–Sato map is located at 0.7. In the simulation, the initial configuration of $f_n(x)$ was given by

$$f_0(x) = \begin{cases} 1 - \dfrac{2}{(1-\epsilon)a} \left| x - \dfrac{(1-\epsilon)a}{2} \right|, & x \in [0, (1-\epsilon)a), \\ 1, & x \in [(1-\epsilon)a, a) \cup \{1\}, \\ 0, & x \in [a, 1). \end{cases} \tag{8}$$

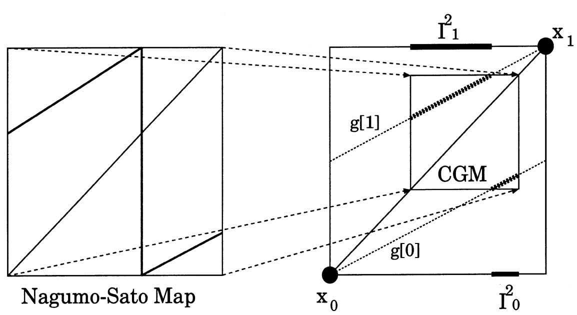


Fig. 4. An embedding of the Nagumo–Sato map. A transformation multiplying ϵ and moving $(1 - w)$ along the x- and $y = f_n(x)$-directions embeds a map into $g(x)$ for $\epsilon = 1 - k$.

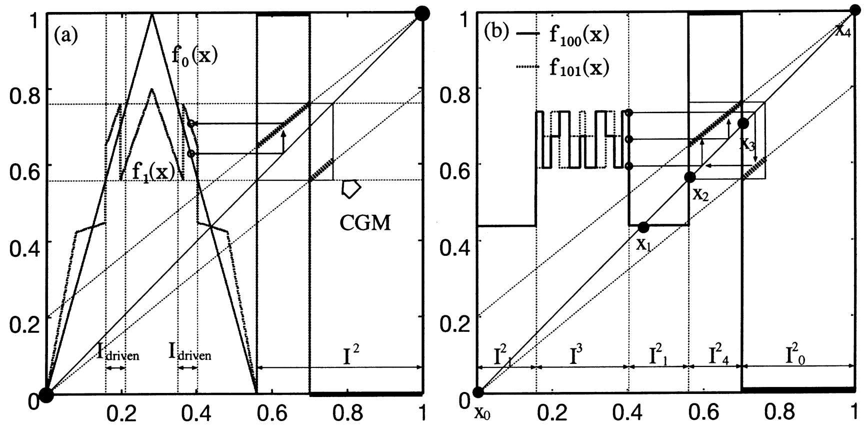


Fig. 5. Time evolution of f_n. The function f_n is divided into two parts, the map area and the rest. Here, fixed intervals produce the Nagumo–Sato map, and the dynamics of $f_n(x)$ which is mapped to the square is determined by the map. Here, $a = 0.7$, and $\epsilon = 0.2$. At the square, the generated map $g|_{I_{\rm map}}$ has the same properties as the map $x_{n+1} = 0.8x_n + 0.44$. (a) The graph of f_0 and f_1 are plotted. Here, I^2 and $I_{\rm driven}$ for $n = 0$ are displayed. f_0 is chosen as described in the text. The dynamics of the function mapped to the square region is determined by the generated map, while the remaining part converges to a type-II fixed point. (b) f_{100} and f_{101} are plotted. All points converge to fixed or periodic points. The periodic points are determined by the Nagumo–Sato map. Each point is period-3, and as a whole f_n is a period-3 function.

With the evolution of our functional dynamics, the function $f|_{I_0^2\cup I_1^2}$ determines the Nagumo–Sato map. The remaining part ($I_{\rm rest}$) of the interval (i.e., which is mapped according to a distorted tent map) folds by itself (see I) and if it is mapped to a value in I_0^2 or I_1^2, it subsequently evolves under the generated Nagumo–Sato map. Fig. 5(b) shows snapshots of the function $f_n(x)$ for $n = 100, 101$. The function converges to a periodic function as a whole. The period of the cycle is derived from the generated map. The functional values of two different points x' and x'' having the common periodic cycle changes synchronously ($f_n(x') = f_n(x'')$), because the difference in $f_n(x)$ values decreases during the transient process before $f_n(x')$ and $f_n(x'')$ are attracted to the periodic motion, and also the Nagumo–Sato map has a contraction property (with slope less than 1) in each branch. As $n \to \infty$, the points in the interval I are contained in either I^1, I^2 or I^3, and $I_{\rm rest}$ vanishes.

In general, an f_n with (at least) two type-I fixed points has a potential to possess a Nagumo–Sato map as a generated map. To consider a general situation with multiple type-I points with several type-II intervals, we define the 'multi-branch Nagumo–Sato map' by (6). In this case, the graph of the g has the same slope $(1-\epsilon) < 1$ for all x. This type of map can be generated generally from random initial conditions. In fact, in the inset in Fig. 1(b), the graph of g consists of several branches with the same slope $1-\epsilon$.

With the multi-branch Nagumo–Sato map, a function f_n periodic in n with an arbitrary period can exist for all ϵ. We denote a value of a type-III point $f_n(x')$ as a_n. If $g|_{I_{\rm map}}$ with a period-m attractor is given, the set of values of the type-III point is determined as $\{a_i|a_{i+1} = g(a_i), i \bmod m\}$. Then a new attractor with period-$(m+1)$ is obtained by choosing an initial function to have two new branches properly. We can arrange branches $[i]$ and $[j]$ to satisfy the conditions that $g_n[i](a_{m-1}) = a_m$ and $g_n[j](a_m) = a_0$ for an arbitrary periodic orbit. [4]

By choosing initial functions suitably, we can have rather complex dynamics based on the multi-branch Nagumo–Sato map. In Appendix B, the coexistence of multiple attractors is demonstrated, while it is also shown that $g|_{I_{\rm map}}$ can have countably infinite attractors by suitably choosing the initial conditions to generate the multi-branch Nagumo–Sato map.

[4] There is some restriction on a_m so that x_i and x_j cannot be the same.

In the argument above, the function g is defined at a countable number of points of x (i.e., the attractor of the generated map $x_{n+1} = g(x_n)$ has a measure zero basin). However, if all type-II fixed points are in a connected type-II fixed interval, each attractor has a finite measure basin. When f_0 is a random function, such connected intervals are formed. In fact, a multi-branch Nagumo–Sato map is often generated from random or other initial functions. In general, the width of each branch is not identical, and a complicated combination of branches is generated. As shown in I, intervals of type-II fixed points form a fractal structure. Hence, branches in the generated map have an infinite number of segments with a fractal configuration in general. Thus $f_n(x)$ can evolve with a complicated cycle that may be of infinite period.

3.3. Case with finite type-I fixed intervals

In the cases considered to this point, the generated map in the functional dynamics (1) cannot exhibit chaotic instability, in the sense that the slope of the map is less than 1 for almost all points. Except for a set of discontinuous points, all generated maps have a slope $1 - \epsilon$. Here we study how a generated map can have a larger class of one-dimensional maps that allow for chaotic instability.

To study this class of functional dynamics, we extend our consideration to the case with a connected interval of type-I fixed points, i.e., with an interval of type-I fixed points ($f(x') = x'$ for all $x' \in I_i^1$). The existence of such an interval is exceptional in this functional map system, in the sense that it is almost impossible to produce such an interval by the evolution (1) unless the initial function does not include such an interval. Indeed, a monotonically increasing function converges to a step function, and a single-humped function tends to converge to a function consisting of isolated type-I fixed points and connected intervals of type-II fixed points [1].

Although an initial function evolving into a function possessing type-I fixed intervals is rather rare in functional space, there are some reasons to study the situation: (1) such an initial function may have some meaning in our model (see the discussion in Section 5) and (2) choice of type-I fixed intervals is convenient to study the hierarchy of meta-maps, to be discussed in the next section. Accordingly, we assume the existence of type-I fixed intervals.

Then, we define sets of type-II fixed points in the same way as in the last section. The type-I fixed intervals are labeled as $I_0^1, I_2^1, \ldots, I_{m-1}^1$. Now, I_i^2 is defined as an interval where $f|_{I_i^2} \subset I_i^1$ (see Fig. 6). Although in the last

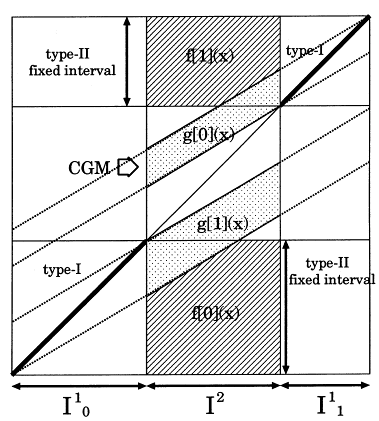


Fig. 6. The closed one-dimensional map. Using two type-I fixed intervals, we can obtain a larger class of a one-dimensional map. The type-II fixed function $f(x)$ is represented by $f[0](x)$ (for $x \in I_0^1$) and $f[1](x)$ (for $x \in I_1^1$). I^2 is a domain of the type-II fixed function $f(x)$. The generated map $g(x)$ is given by $g[0](x) = (1-\epsilon)x + \epsilon f_0(x)$ (for $x \in I_0^1$) and $g[1](x) = (1-\epsilon)x + \epsilon f_1(x)$ (for $x \in I_1^1$). We call this type CGM a 'unit-I'. In this figure $\epsilon = 0.5$.

section, we considered the case in which f_n is a constant function over an interval I_i^2, in the present case, each $f|_{I_i^2}(x)$ can have various values in the interval I_i^1. Let us write $f|_{I_i^2}$ as $f[i]$. The generated map corresponding to the fixed function $f[i](x)$ has the form

$$g|_{I_i^2}(x) \equiv g[i](x) = (1-\epsilon)(x - f[i](x)) + f[i](x), \quad x \in I_i^2, \quad f[i](I_i^2) \subset I_i^1. \tag{9}$$

This function $g[i](x)$ is bounded both from above and below, because $f[i](x)$ has a possible minimum value $\min I_i^1 = a$ and possible maximum value $\max I_i^1 = b$, $(1-\epsilon)(x-a) + a \le g[i](x) \le (1-\epsilon)(x-b) + b$ (see Fig. 6). It is natural to call this function $g[i](x)$ within this bounded area the 'i-branch', in analogy to the last section. For each type-I fixed point x_i, the generated map is given by $(1-\epsilon)(x - x_i) + x_i$, although x_i here can change continuously.

Consider the union of m type-I fixed intervals $I^1 = I_0^1 \cup I_2^1 \cup \cdots \cup I_{m-1}^1$. If type-II fixed intervals are within an interval ($\min I^1$, $\max I^1$) and the condition $g|_{I^1 \cup I^2} \subset I^1 \cup I^2$ is satisfied, the generated map on the interval $I^1 \cup I^2$ is a CGM (the corresponding $f|_{I^1 \cup I^2}$ is unit-I). Here, in a type-II fixed interval corresponding to a type-I fixed interval, the generated map $g(x) = (1-\epsilon)x + \epsilon f(x)$ no longer has a constant slope. Rather the slope $g'(x) = (1-\epsilon) + \epsilon f'(x)$ varies with x.

Following the argument in the last section, we start from the case with two type-I intervals. Now, we divide the interval I into a type-I fixed interval I^1 and a type-II fixed interval I^2. Then, the interval I^1 is divided into two parts, I_0^1 and I_1^1. Without loss of generality, we can take $\min I_0^1 = 0$ and $\max I_1^1 = 1$. The fixed partial function consisting of type-II fixed points is determined as $f[0](x) \in I_0^1$ for $x \in I_0^2$ or $f[1](x) \in I_1^1$ for $x \in I_1^2$. Then the generated map $g(x)$ is given by

$$g(x) = \begin{cases} x, & x \in I^1, \\ g[0](x) = (1-\epsilon)x + \epsilon f[0](x), & x \in I_0^2, \\ g[1](x) = (1-\epsilon)x + \epsilon f[1](x), & x \in I_1^2. \end{cases} \tag{10}$$

The area where the graph of the generated map can exist is denoted by the dotted area in Fig. 6. Any one-dimensional map included within the dotted area can be embedded into our functional map by choosing the configuration of the type-II fixed function within the shadowed area. Since any function can be embedded in the dotted area, it is possible to have a case with $|g'(x)| > 1$. Indeed, we will give an explicit example satisfying $|g'(x)| > 1$ in Eq. (14).

In Fig. 6, there are intervals where $g(x) \in I^1$. If the generated map exists in such a region, a point evolving as a type-III point may be absorbed into this region and become a type-II point. Indeed, when the point is mapped into this region, $f_{n+1}(x') \in I^1$ is satisfied, and the point x' becomes a type-II fixed point. On the other hand, if $g(x) \in I^2$, a type-III point remains a type-III during the entire evolution, and it never becomes a type-II fixed point.

Is there some restriction on the possible form of a generated map allowed by the present functional dynamics? As discussed in Appendix C, there is some restriction according to the present embedding of the generated map. However, as is also discussed in that appendix, an arbitrary one-dimensional map can be embedded as a generated map by considering a two-step iteration, i.e., as a map to determine $f_{n+2}(x)$ from $f_n(x)$.

4. Meta-map in functional dynamics

In Section 3, we have shown how a one-dimensional generated map is formed by a suitable configuration of type-I and type-II fixed points. In the example, the one-dimensional map is explicitly constructed with the condition

$g_{I^1 \cup I^2} \subset I^1 \cup I^2$. We call the partial function $f|_{I^1 \cup I^2} \equiv f|_{U^1}$ a 'unit-I'. In this case the generated map $g_n|_{I^1 \cup I^2}$ is fixed in time. However, the CGM condition ($g_n(J) \subset J$) does not necessarily impose the condition for the 'unit-I'. Then, $g_n|_J$ is not necessarily a fixed function. In this section, we consider such case in which a generated map changes dynamically in time.

In the situation discussed in Section 3, in order for there to exists a generated map to determine the dynamics of the type-III points, it is essential that the map stays within a bounded area. The type-I fixed intervals determine where type-II fixed points can exist, from which the type-II fixed function never leaves. The configuration of type-I and type-II fixed points determines bounded areas in which the generated map remains as a branch. The type-II fixed function determines a generated map within the bounded areas (see Fig. 6).

In the last section, we considered the situation in which the dynamics of the type-III point determined by $g|_{I^1 \cup I^2}$ evolves within the interval $I^1 \cup I^2$, according to the type-II and corresponding type-I fixed points. This process can be extended hierarchically. In this section, we consider a unit-I instead of a type-I fixed interval and a type-III point instead of a type-II point, to see the dynamics of $f_n(x)$ determined by the type-III point.

The unit-I ($f|_{U^1}$) determines an interval where type-II and type-III points can exist. The domain of the partial function $f|_{U^1}$ is $I^1 \cup I^2$. Thus, $f_n|_{I^2 \cup I^3} \subset U^1$ consists entirely of type-II fixed points and type-III points. Here, the dynamics of the type-III points are determined by a CGM, and their motion is confined within this region. Thus, we replace the type-I fixed interval with unit-I by the transformation (5) (see Fig. 10(a) and (b)). In case considered in the last section, the configuration of type-I fixed intervals determines where the branches exist. Here, the arrangement of unit-I determines where the branches of the generated map exist.

First, we elucidate the branch structure determined by a unit-I (see Fig. 7). These branches are derived from type-I and type-II points. As noted at the branch derived from a type-I point, a one-dimensional map is given according to the configuration of the type-II fixed function. In the same way, at a branch derived from type-II points, a bounded map $g_n(x)$ exists according to the configuration of the type-III function. Since $f_n|_{I^3}$, consisting of the type-III

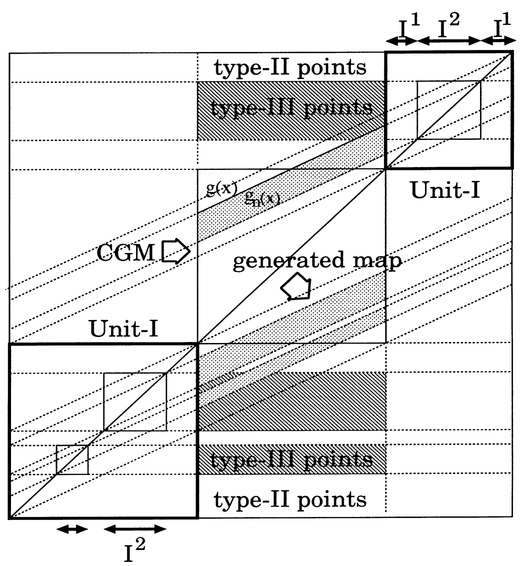


Fig. 7. The hierarchical arrangement of a 'meta-map'. The shaded area is a region where type-III points can exist and the dotted branch indicates a region where the generated map derived from type-III points can exist. A generated map in I^3 has an n-dependence ($g_n|_{I^3}$).

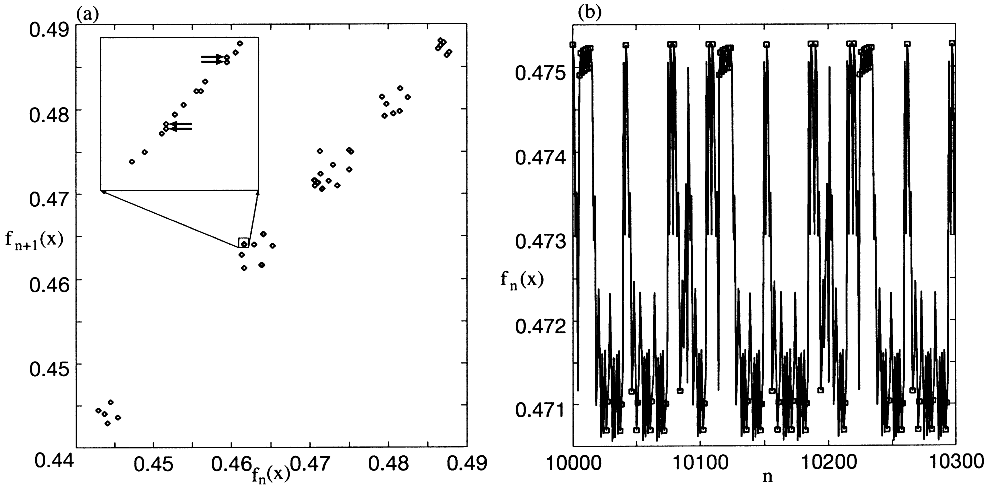


Fig. 8. (a) Another close up of the return map of Fig. 1(a). At the points x' indicated by arrows, $g_n(x')$ has two values. (b) Time evolution of $f_n(x')$ for $10\,000 < n < 10\,300$, which is determined by the meta-map and has a period 111. The function $f_n(x')$ is a type-IV point at the steps n plotted with squares, while it is type-III otherwise.

points, depends on n, the generated map, $g_n(x) = (1-\epsilon)x + \epsilon f_n(x)$, also depends on n:

$$
\begin{aligned}
&f|_{I^1 \cup I^2}(x), \qquad f(x) \in I^1, \\
&g|_{I^1 \cup I^2}(x) = (1-\epsilon)x + \epsilon f(x), \\
&f_n|_{I^3}(x) = g(f_{n-1}(x)), \qquad f_n(x) \in I^2, \\
&g_n|_{I^3}(x) = (1-\epsilon)x + \epsilon f_n(x).
\end{aligned} \tag{11}
$$

Here, $g_n(x)$ is determined from the type-III function and change with time step n. The point $f_n(x') \in I^3$ evolves according to $g_n(x)(f_{n+1}(x') = g_n(f_n(x')))$. Since $g_n(x)$ is not a fixed function, it represents a change of rules. Accordingly, we call this type of map a 'meta-map'. By using these branches, we can construct a new CGM consisting of $g(x)$ and $g_n(x)$. The type of point x, which evolves under the CGM can change in time. The interval I can be written $I^1 \cup I^2 \cup I_n^3 \cup I_n^4 \cup I_{\text{rest}}$. Now, I^3 and I^4 have the suffix n.

When the dynamics of type-III points are periodic, determined by a (multi-branch) Nagumo–Sato map, the dynamics of a type-IV point determined by the type-III points is also periodic. Indeed, this hierarchical structure is often formed starting from a random initial function, since a generated map of the Nagumo–Sato type is commonly formed as mentioned in Section 3. In Fig. 8(a), an example of a meta-map (return map) is plotted. These data were obtained with a numerical simulation starting from a random initial function (see Section 3.1) f_0. For the points indicated by arrows, the return map has two values. Hence the dynamics of the points are not determined by a fixed generated map, but by a time-dependent generated map. In this case, the 'type' of a point x' is no longer fixed, but can change between type-IV and type-III, depending on the intervals in which $f_n(x')$ is located, as n changes. The evolution of the 'type' of a particular x' is plotted in Fig. 8(b).

A simple example of the 'type' change is displayed in Fig. 9. Here, fixed points in the right-hand part determine $g|_{I^2}$, which determines the dynamics of type-III points with period-2. The type-III points determine a time-dependent map that switches between $g_{\text{odd}}|_{I^3}$ and $g_{\text{even}}|_{I^3}$. The fixed points on the left-hand side determine $g[0](x)$, which determines the dynamics of the type-III points. Here, the fixed map consisting of $g[0](x)$ and $g_{\text{even}}|_{I^3}$ generates a period-2 orbit. If the evolution of the $f_n(x')$ is determined by $g_{\text{even}}|_{I^3}$ at even n and $g[0](x)$ at odd n, $f_n(x')$ changes

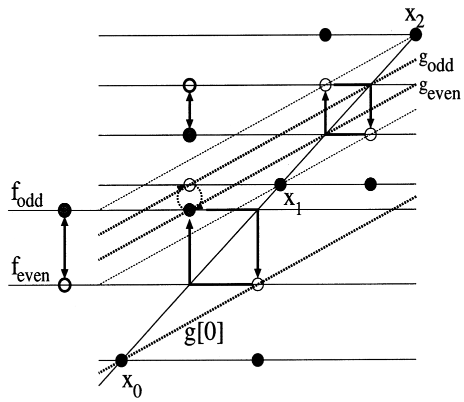


Fig. 9. One of the simplest configurations for 'type' change. Type-I fixed points x_1, x_2 and two type-II points determine a period-2 map (as in the upper-right square). A period-2 motion driven by the generated map determines a period-2 branch of $g_n(x)$. Then, $g[0](x)$ and $g_{\text{even}}(x)$ produce a period-2 orbit (as in the lower left square). If the evolution of a point $f(x')$ is determined by $g[0](x)$ (or $g_{\text{even}}(x)$), $f(x')$ is a type-III point (or type-IV point, respectively). The sequence of the 'type' of $f(x')$ is III, IV, III, IV, III, ...

cyclically with period-2 as

$$f_{\text{even}}(x') = g[0](f_{\text{odd}}(x')), \qquad f_{\text{odd}}(x') = g_{\text{even}}(f_{\text{even}}(x')), \tag{12}$$

and the 'type' of the point x' also changes cyclically with period-2 as III, IV, III, IV, ... On the other hand, if the evolution of $f_n(x')$ is determined by $g_{\text{even}}(x)$ at odd n or $g[0](x)$ at even n, $f_n(x')$ is a type-II fixed point.

When a type-III point possesses a chaotic orbit, as given in Section 3.3, the nature of the functional dynamics determined by this type-III point is more interesting. Let us study the case with a chaotic generated map by constructing an example. By choosing a suitable initial function, one can embed a one-dimensional map to construct a meta-map explicitly. For example, we adopt the following one-dimensional map to be embedded:

$$x_{n+1} = 2(1-\epsilon)x_n + \epsilon \pmod 1. \tag{13}$$

This map has chaotic orbits for $\epsilon < \frac{1}{2}$. Indeed, the initial function f_0,

$$f_0(x) = \begin{cases} x, & x \in [0, \frac{1}{2}(1-\epsilon)], [\frac{1}{2}(1+\epsilon), 1.0], \\ \dfrac{1-\epsilon}{\epsilon}x + \dfrac{3\epsilon - 1}{2\epsilon}, & x \in (\frac{1}{2}(1-\epsilon), \frac{1}{2}], \\ \dfrac{1-\epsilon}{\epsilon}x - \dfrac{1-\epsilon}{2\epsilon}, & x \in (\frac{1}{2}, \frac{1}{2}(1+\epsilon)), \end{cases} \tag{14}$$

determines this map. In Fig. 10(a), the graph of the f_0 is plotted for $\epsilon = \frac{1}{3}$. The dotted line represents the graph of the g_0 that has exactly the form $x_{n+1} = 2(1-\epsilon)x_n + \epsilon$.

To construct a meta-map, we replace the type-I fixed interval in Fig. 10(b) with this f_0 (unit-I) by the transformation (5). With this nested structure, there are intervals where a type-III function exists, and the function determines an n-dependent $g_n(x)$ (CGM). This CGM acts as the map for a point that satisfies $f_n(x') \in I^1 \cup I^2 \cup I^3$. For this configuration, the dynamics of a part of a CGM are determined by type-III points. In this case, a point which satisfies $f_n(x') \subset I^3$ behaves as a type-IV point under the iteration.

In Fig. 11(a)–(e), the evolutions of $f_n|_{\cup_{i=1}^3 I^i}$ and the meta-map are plotted, where the map $g_n|_{\cup_{i=1}^3 I^i}$ is plotted as a dotted line. Type-III points evolve according to $f_{n+1}(x') = 2(1-\epsilon)f_n(x') + \epsilon$. In the present example, the

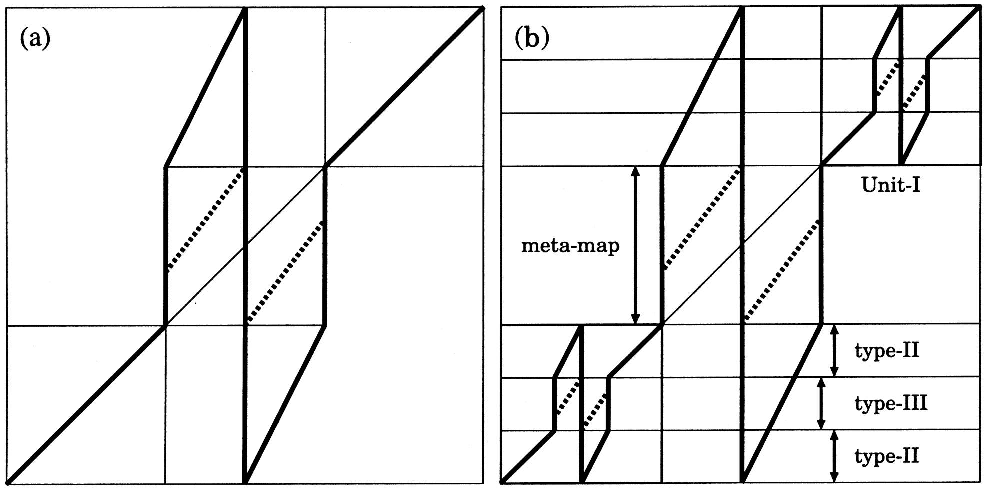


Fig. 10. (a) A configuration (indicated by the solid line) which determines a generated map $x_{n+1} = 2(1-\epsilon)x_n + \epsilon (\mathrm{mod}\, 1)$ (indicated by the dotted line) in the center square ($\epsilon = \frac{1}{3}$). (b) An initial function f_0 leading to a hierarchical configuration of the map (a).

slope $g'(x) = 2(1-\epsilon)$ is constant, and the slope of the type-III function $f_n|_{I^3}$ is easily calculated as: $f_n'(x) = ((1-\epsilon)/\epsilon)2^n(1-\epsilon)^n$. The gradient of the generated map is determined as $g_n'(x) = (1-\epsilon) + \epsilon f_n'(x)$, and it has the form $g_n'(x) = (1-\epsilon)\{1 + 2^n(1-\epsilon)^n\}$. Hence, this meta-map has a part where its gradient increases exponentially with n.

This implies that our functional dynamics can have stronger orbital instability than deterministic chaos: a tiny deviation δ from a point mapped to this type-III points grows as $\prod_{n=i}^{N}|g_n'(x)|$. Since $\prod_{k=1}^{n}\alpha^k = \alpha^{n(n+1)/2}$, the leading order of the exponent of the orbital instability is n^2. Hence, the orbital instability is such that a tiny deviation grows as exp(const. $\times\, n^2$) rather than exp(const. $\times\, n$) as is the case in conventional chaos. Due to this strong instability based on chaotic dynamics in the generated map, we call this dynamics 'meta-chaos'. In Fig. 11(f), an example of the orbits for meta-chaos is displayed. This evolution is determined by $g_n(x)$. The 'type' of the point changes between III and IV according to the map $g_n(x)$.

For a numerical simulation with this meta-chaos, the required mesh size increases as 2^n. Hence, a simulation quickly becomes invalid as n increases.

In the example mentioned above, we have constructed a meta-map by choosing special initial conditions. However, we note again that a meta-map configuration itself is not special and can be reached, for example, from a random initial function. Still, it is very rare to obtain a connected type-I fixed interval from random initial conditions. Hence, in most simulations from arbitrarily chosen initial functions, we mostly observe generated maps of the Nagumo–Sato type, where magnitude of the slope, $|g_n'(x)|$, is always less than 1.

The nesting process of the meta-map can be continued hierarchically, since the configuration of type-I, type-II and type-III points, discussed above, can be a CS, in which the generated map becomes a CGM as a whole. This arrangement of $f_n|_{U^2} \equiv f_n|_{\bigcup_{i=1}^{3} I^i}$ is called a 'unit-II'. One can replace a unit-I in the above construction by such a unit-II. In such a situation, type-IV points determine a generated map, and with an appropriate configuration, the generated map can be a CGM. Now we can call the partial function $f_n|_{U^3} \equiv f_n|_{\bigcup_{i=1}^{4} I^i}$ a 'unit-III'. This hierarchy to form a 'unit-N' can be continued for $N \to \infty$ (see Fig. 12). To continue this nesting process, we define the 'unit-N' and the Nth level meta-map as follows.

A unit-N is a partial function $f_n|_{U^N}$. U^N consists of type-I, II, ..., $N+1$ points and satisfies the condition $g_n|_{U^N} \subset U^N$. A point $f_n(x') \in U^N$ (with $x' \notin U^N$) evolves by the unit-N and has a 'type' from II to $N+2$. We

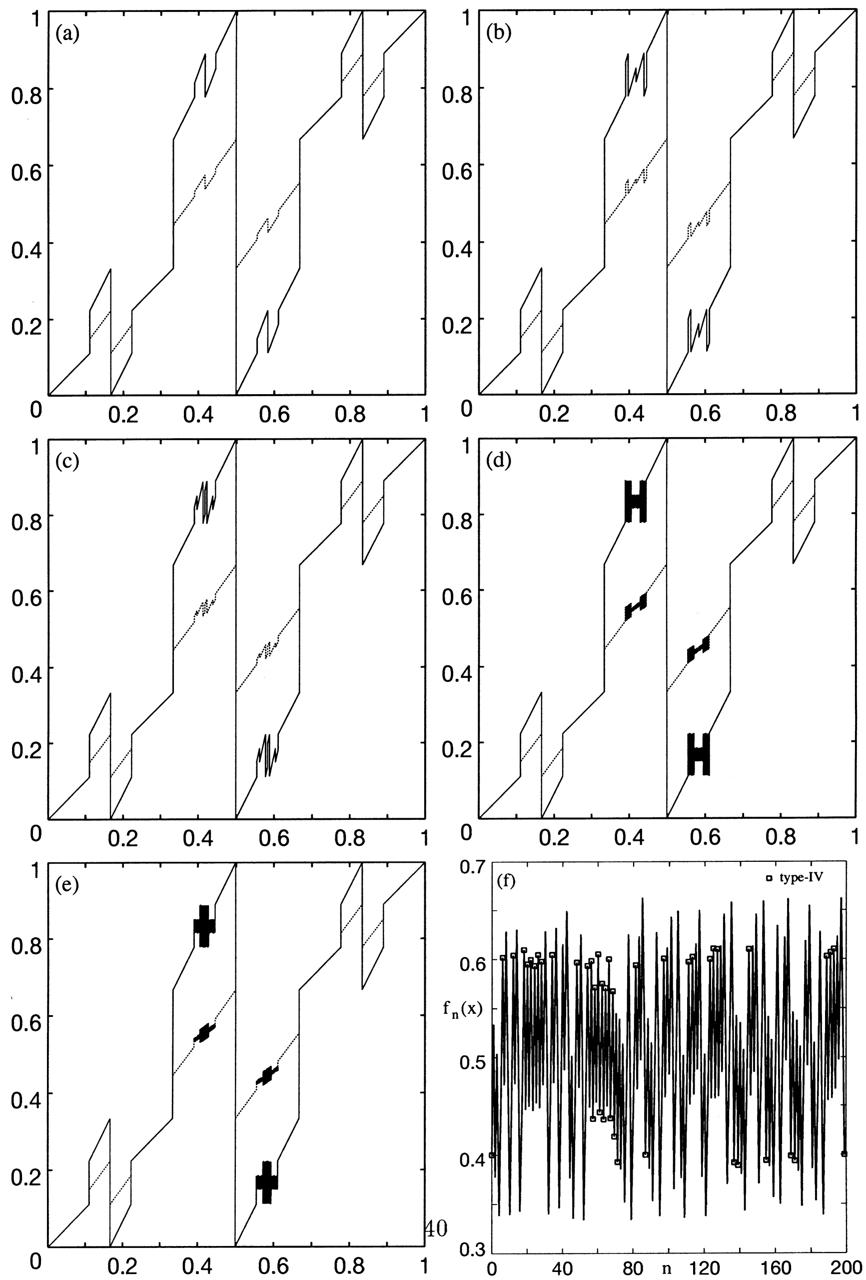


Fig. 11. Time evolution of $f_n|_{\cup_{i=1}^3 I_i}$ and $g_n|_{\cup_{i=1}^3 I_i}$: (a) $f_1|_{\cup_{i=1}^3 I_i}$ (solid line) and $g_1|_{\cup_{i=1}^3 I_i}$ (dotted line); (b) $n = 2$; (c) $n = 3$; (d) $n = 10$; (e) $n = 11\, f_n|_{\cup_{i=1}^3 I_i}$ consists of the type-II function and the type-III function. The type-IV function has a shape H in (d) or + in (e). (f) Time evolution of one point $f(x')$, $x \in I \setminus \cup_{i=1}^3 I_i$ determined by the generated meta-map, at the center square ($0 < n < 200$). The 'type' of the function $f_n(x')$ changes between III (driven by the piecewise linear part of $g(x)$) and IV (driven by time dependent $g_n(x)$ (H, +)) in time. Squares indicate that the 'type' of $f_n(x')$ is IV at n. Since the mesh size required for the computation is extremely large, the plotted orbit is not precise. It is expected, however, that the statistical properties are conserved with this numerical computation.

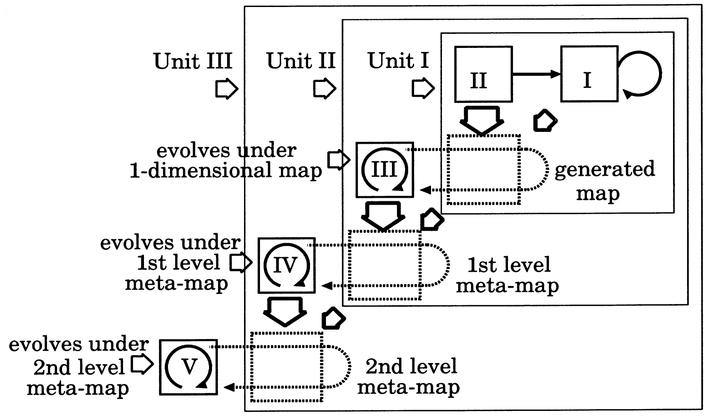


Fig. 12. The schema of the meta-map. A one-dimensional map can be determined from the configuration of type-I and type-II fixed points (unit-I). The function $f_n(x)$ iterated by this map is a type-III point which determines a meta-map (unit-II). The meta-map is determined by the map determined from a type-II fixed function and determines a dynamics of a type-IV point. The type-IV points and unit-II determine a higher level meta-map. This process can be continued recursively.

denote a function defined on an interval of type-N points as $f_n^N(x)$ and the generated map determined by $f_n^N(x)$ as $g_n^N(x)$ (a fixed function is written by $f(x)$ instead of $f^{\mathrm{II}}(x)$). The functional equation rewritten in a recursive form with respect to the 'type' has the form

$$f_{n+1}^M(x) = g_n^{N-1}(f_n^N(x)), \quad x \in I_n^N, \quad f_n(x) \in I_n^{N-1}, \qquad g_n^N(x) = (1-\epsilon)x + \epsilon f_n^N(x), \quad x \in I_n^N. \tag{15}$$

The Nth level meta-map is defined as a CGM consisting of $g|_{I^1 \cup I^2}, g_n^{\mathrm{III}}|_{I_n^3}, \dots, g_n^{N+2}|_{I_n^{N+2}}$, where a one-dimensional map determined from type-I and type-II points is called the '0th level' meta-map. All meta-maps depend on the fixed function $f|_{I^1 \cup I^2}$ and are constructed recursively as $f|_{I^1 \cup I^2}, f_n^{\mathrm{III}}|_{I_n^3}, \dots, f_n^N|_{I_n^N}$. The whole interval I can be written $I^1 \cup I^2 \cup I_n^3 \cup \dots \cup I_n^N \cup \cdots$. Here note that a 'type' greater than 2 can change in time, although each point has a finite maximal value of its 'type', depending on the initial configuration.

The Nth level meta-map is determined by the configuration of type-I, II, ..., $N+2$ points. It is important that each unit-N and each branch are bounded. We can arbitrarily arrange any unit-N and type-III, IV, ..., $N+2$ points according to the branches. The configuration producing a meta-map characterizes a 'syntax' for each x. Each x has a time evolution as a type. The 'type' of a point that is of type-III or higher changes in time. For a meta-map higher than second level, there is a sequence, for example, III, III, IV, V, III, ... There is a transition relation among type-N ($N > 2$) points. Each point evolves under a hierarchy of meta-maps. In the above representation, the dynamics of the Nth level meta-map is independent of that of the type-$N+3$. This means that if noise is added to $f_n(x')$, the effect of the noise spreads from the lower-level unit-N to the higher-level unit-$M (M > N)$.[5]

In a high level meta-map with the type-I fixed intervals, the orbital instability is stronger than the exponential instability of conventional chaos. If $g'(x) \sim \alpha (|\alpha| > 1)$ and $f_i^{\mathrm{III}}(x)$ is not a constant function (i.e., has a gradient $\beta \neq 0$), then $f_n^{\mathrm{III}'}(x) \sim \beta \prod_{k=i}^n \alpha \sim \alpha^n$, and $g_n^{\mathrm{III}'}(x) \sim \alpha^n$. Now, the leading order of the slope of the first-level

[5] In this section, we have constructed the initial function f_0 explicitly and studied the possibility of the existence of the meta-map. Here the hierarchy of unit-N is constructed step by step and the change of the types of a point x' are restricted by the hierarchical combination of unit-N. In the general case, the dynamics of the f_∞ is not always described as the evolution of points following a hierarchy of meta-maps, even if the type of each point is defined. There is a change of types which breaks the hierarchy of meta-map. In such case, a partial function $f_n|_A$ evolves under a partial generated map $g_n|_B$ determined by a partial function $f_n|_B$, then at the next step $f_{n+1}|_B$ evolves under $g_{n+1}|_A$ even after the transients are decayed out (i.e., $n \to \infty$). In other words, there is a set of initial function to be divided into partial functions which determine the dynamics each other in turns. Such a 'dynamical change of types' will be studied in our future paper.

meta-map is α^n as is mentioned above. A type-IV function $f_n^{\mathrm{IV}}(x)$ evolves under this first level meta-map. If $f_i^{\mathrm{IV}}(x)$ is not a constant function and has a gradient $\gamma \neq 0$, $f_n^{\mathrm{IV}'}(x)$ is calculated as $\gamma\prod_{k=i}^{n}\alpha^k \sim \alpha^{n^2}$, and $g_n^{\mathrm{IV}'}(x) \sim \alpha^{n^2}$. Hence, $f_n^{\mathrm{V}'}(x) \sim \alpha^{n^3}$ and $g_n^{\mathrm{V}'}(x) \sim \alpha^{n^3}$. Repeating this argument, the leading order of the slope of the Nth level meta-map is given by α^{n^N}. Thus a tiny deviation from a point, which evolves under the meta-map, is amplified by $|\alpha|^{n^N}$ at each n step. Because of this, an Nth level meta-map has an orbital instability that behaves as $\exp(\text{const.} \times n^{N+1})$. The level of the orbital instability increases with the level as $\exp(\text{const.} \times n^{N+1})$. In other words, an exponent λ corresponding to the Lyapunov exponent of conventional chaos increases as $\lambda \sim n^N$ as n increases for the Nth level meta-map.

5. Summary and discussion

In the present paper, we have studied functional dynamics, focusing on the generation of rules (mappings) for the dynamics representing change of a function, and on the hierarchy of meta-rules.

As a first step, we introduced a new concept, the ‘generated map’ g_n, which is given from f_n and determines the dynamics of f_n. The dynamics of some other parts of x are determined by this generated map, while a CGM is defined as one that maps a region into itself. Functional values on some intervals were shown to change according to the generated map. This leads to a one-dimensional map or a ‘meta-map’ that changes the map itself.

In Section 3, we explicitly showed that some classes of one-dimensional maps are embedded into this functional dynamics. In Sections 3.1 and 3.2, a piecewise linear map with two intervals of the slope $1 - \epsilon$ were shown to be generated from two type-I fixed points and two intervals of corresponding type-II fixed points. Next, this construction was generalized to cover the case with several isolated type-I fixed points and the corresponding type-II intervals. There, a piecewise linear map with several intervals with slope $1 - \epsilon$ were found to be generated. This map, called a ‘multi-branch Nagumo–Sato map’ exhibits periodic cycles. Hence, the dynamics of the functional values determined by this generated map display a periodic cycle, which explains why periodic cycles are often generated in our functional dynamics.

In Section 3.3, generated maps with type-I fixed intervals and type-II points were discussed. In this case, a one-dimensional map with an arbitrary slope can be embedded. Now, the functional dynamics determined by this generated map can also exhibit chaotic dynamics.

As shown in Section 4, this construction of generated maps can continue hierarchically. The dynamics determined by a generated map forms a higher-level generated map that determines the dynamics of other regions. Since this map is changed by the first generated map, it is regarded as a ‘meta-map’, a map determined by another map. This procedure can be continued ad infinitum, leading to meta–meta ... maps. When a generated map exhibits chaotic dynamics, as discussed in Section 3.3, the dynamics by meta-map can exhibit ‘meta-chaos’, in the sense that the evolution rule itself changes chaotically in time. It was shown that this meta-chaos has a stronger orbital instability than in chaos, in the sense that a small deviation is amplified as $\exp(\text{const.} \times n^{M+1})$ for the Mth level meta-map, rather than $\exp(\text{const.} \times n)$.

Now, we discuss some relevance of our results for the target problems listed in Section 1. Eq. (1) represents a process of iterating a function by referring the function itself. This iteration is introduced to study the change of abstract input–output network. Our cognition process to generate language is thought to depend on iteration of input–output relationships, and dynamic change of such relationships. External inputs that are inarticulated are processed in our cognitive process recursively, and some symbols and rules to process them are formed. Hence it is important to study a minimal model that captures iterative process to change input–output relationship. Our functional dynamics give such model and we can extract a minimal mechanism to separate rules and objects from inarticulated closed system. Although our model may not directly correspond to some specific cognitive process in our brain, it can capture the essence of how the articulation and rule generation are generally possible in a feedback process of input–output relationship.

The basic structure of the functional dynamics is provided by two types of fixed points. They are invariant under iterations and determine a fixed generated map.

The invariant parts of the function consists of type-I and type-II functions, where the self-reference relation $f(x) = f \circ f(x)$ is satisfied. In general, f_0 has some points that fall on to fixed points of the iteration. Then, type-I points and type-II functions are formed through the iteration process (see also I). Through the iteration of the functional dynamics, the invariant part grows. This process is called articulation process in I.

The invariant part determines a fixed generated map which drives type-III points. This means that if the fixed points are removed by perturbing the initial function slightly on $I^1 \cup I^2$, the dynamics of type-III points are influenced. On the other hand, even if a type-III point is removed, there is no influence to the dynamics of the invariant part.

Hence, we can regard that the invariant part is elementary than type-$N > 2$ points. In the problem of language, the invariant part corresponds to 'nouns' or basic substances. A type-I point is a point of the filter where accepts an input as it is $(f(x) = x)$ and the type-II point is a point identifying an object with a type-I point. A type-III point is determined from the invariant part and indicates a set $\{f_n^{\mathrm{III}}(x)\}$ at $n \to \infty$. The set is determined from the invariant part and has type-II points as elements. At the same time the orbit of the type-III point determines a sequence of type-II points. By focusing on the aspect of a set to classify type-II points, the type-III points are regarded to represent a categorization (of type-II points), while by noting the aspect of type-III points as an orbit of a sequence of type-II points, they are regarded as an operation over words. In the former viewpoint, the categorization means a noun representing a set of nouns, while in the latter, the operation means an action to connect verbs with nouns to form a sentence.

Similarly, the first-level meta-map determines a set of type-II and type-III points and an orbit consisting of type-III and type-IV points, as an operation on a set of type-II and type-III points. In this hierarchical configuration, each orbit is characterized by a sequence of types and a sequence of values $f_n(x)$. A point, which evolves under the Nth level meta-map, changes its 'types' among type-III, IV, $\dots$, $N + 3$. This hierarchy of types means the hierarchical categorization in classification, in one sense, while the sequence of 'types' provides a basis for the hierarchical structure in grammar, seen, for example in a noun phrase or a relative pronoun.

A map and a meta-map determine an orbit, which evolves following a lower level structure in the hierarchy. In our system, a higher-level structure is formed based on the lower-level structure, which we believe is an important characteristic in language. For example, the cognitive language theory [9] captures the language as a network of words where some words, called prototype, are elementary and other words are arranged in connection with the prototypes. It is important that 'stability of words' against external perturbations is discussed there. The prototype is derived from the restriction of our own body, or from a common feature in our society. For other words that are not a prototype, similar words in a different society can have a different representation in the network. In our functional dynamics, the fixed points appear as the invariant part and plays a central role to construct a network (in fact, if the fixed points for the invariant part are removed, the network is broken). Although the cognitive language theory does not focus dynamical aspects and has studied the static structure of language yet, the method which can deal a structure as the categorization and operation at the same time will be needed. We believe that the present study will provide a tool to study the cognitive process in language, even though the study at present is rather preliminary.

In our system, a hierarchical structure is formed through iterations. As mentioned, this hierarchy is also a characteristic of language, and it is important to note that a simple class of functional dynamics with recursive structure can provide such hierarchy in general. The hierarchical structure in our functional dynamics has strong dependence on the lower-level structure, since the higher-level structure is determined according to which branch of the generated map is taken by the orbit.

The form of the generated map g_n depends on the configuration of type-I fixed points. If they are discrete, the slope of g_n is smaller than 1. When there exists a connected interval of type-I fixed points, g_n can have a slope larger than 1, and the meta-map can have a more complex orbit than in chaos. A connected type-I fixed interval is generated by an identity function over some interval, which corresponds to a filter with which an agent acts in response to the world without interpretation. In other words, chaotic functional dynamics and meta-chaos are generated by adding a continuous input from the external world to the 'closed' world of functional dynamics only with self-reference. In the present model only with self-reference, such an interval with partial identity function is rare to be formed unless we put it as in initial condition, since in the model all the information is given in initial function and no external input is added during the iterations. Here the information of external inputs is restricted only within the initial function, to study how the articulation/rule-formation process continues from f_0. From the study, we can identify what class of initial functions is required to have a certain class of dynamical behavior in function, or to attain a certain class of cognitive structure when the present interpretation of the functional dynamics as cognitive process is possible. Existence of type-I fixed interval as initial condition is one requirement to have meta-chaos, while the hierarchy in types is a general feature of the functional dynamics observed in most initial conditions.

To close our discussion on the language, we make a final speculation. As shown in I, a continuous non-decreasing initial function converges to a fixed function. To have type-III points, f_n needs to have at least two type-I points and two type-II points. If f_0 is a continuous function, there must be at least points satisfying $f_0(x') = f_0(x'') \neq f_0(x''')$ for $x' < x''' < x''$. In other words, two distinct parts of x are assumed to take the same value initially. This initial arrangement corresponds to identifying two distinct objects. Starting from such 'cognitive confusion', the function increases complexity to have a higher-type points, through the iteration. The language may have acquired its complexity starting from such cognitive confusion to identify distinct objects, which probably originates in some restrictions of our body.

Possible extensions of the present study will be discussed in the future. In a two-dimensional version of the functional dynamics, an arbitrary two-dimensional map can be embedded in the same way as in Section 3.3. Because of this, we can embed a Turing machine into this system [10] (see also Appendix B), where the search for a relationship between the generalized shift [10] and meta-dynamics (meta-chaos) will be important.

Non-trivial sets of functions over functions are studied in domain theory [6,7,11]. The most important difference between systems studied in domain theory and our model lies in the dynamical aspects of functions treated only in our approach. However, our meta-map is restricted within some intervals and is not extended over the whole domain. Indeed, in our system the size of the Nth level meta-map decreases with order ϵ^N. However, such a contraction can be removed in a more general functional dynamics. This will be important to obtain functional dynamics allowing for a hierarchy of the meta-map over the whole domain.

Another extension required for language will be the inclusion of dialogue [2,3]. To this point, we have only considered one agent whose function changes recursively. To study the social structure of language, functional dynamics with several agents is necessary.

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Appendix A. Some properties of $F(x, y)$

In this appendix, we investigate a general class of functional maps with the form

$$f_{n+1} = F(f_n, f_n \circ f_n). \tag{A.1}$$

We study a fixed point condition and properties of the generated map.

This type of functional equation has fixed points (fixed functions). First, we define $Z(x)$ from $F(x, y)$. Here, $Z(x)$ is the solution of $x = F(x, Z(x))$. The fixed point condition is defined from $Z(x)$. If the condition $f \circ f(x') = Z(f(x'))$ is satisfied, $f(x')$ is a fixed point. If $F(x, y) = (1-\epsilon)x + \epsilon y$, then $Z(x) = x$, and the fixed point condition is nothing but $f \circ f(x') = f(x')$. The fixed point condition in the present general case is determined as follows. (We give the correspondent equation for the case with $F(x, y) = (1 - \epsilon)x + \epsilon y$ in the square bracket [...] for reference.)

1. If $Z(x)$ is a single-valued function, $f(x) = Z(x)$ is a fixed function over the entire interval ($f \circ f(x') = Z(f(x'))$). [$f(x) = x$ is a fixed function.]
2. The point where $Z(x)$ intersects the identity function ($x' = Z(x') = f(x')$) is a fixed point ($f \circ f(x') = f(x') = Z(x') = Z(f(x'))$). [Type-I fixed point condition.]
3. If a point $(x', f(x'))$ is a fixed point ($f(x') = Z(x')$), a point $(x'', f(x''))$ which satisfies $f(x'') = Z(x'') = x'$ is also a fixed point ($f \circ f(x'') = f(x') = Z(x') = Z(f(x''))$). [There is no such fixed point corresponding to this case.]
4. If a point $(x', f(x'))$ is a fixed point, a point x'' with $f(x'') = f(x')$ is also a fixed point. [Type-II fixed point.]

The most noteworthy difference from the case with $F(x, y) = (1 - \epsilon)x + \epsilon y$ is seen in (3). For a point $f(x'') = Z(x'')$, the fixed point condition is that $Z \circ Z(x')$ is a fixed point. There, $Z(x)$ decides a fixed point condition as an orbit of a one-dimensional map. In other words, the 'attractor' of $Z(x)$ is a fixed point of Eq. (15), and a sequence $\{f(x'), Z \circ f(x') = f(f(x')), Z^2 \circ f(x') = f(Z \circ f(x')), \ldots, Z^\infty \circ f(x') \in \text{attractor}\}$ consists of fixed points.

The functional equation can be divided into

$$f_{n+1}(x) = g_n \circ f_n(x), \qquad g_n(x) = F(x, f_n(x)), \tag{A.2}$$

as in the case $F(x, y) = (1 - \epsilon)x + \epsilon y$. The generated map viewpoint is also effective in this general case.

However, for a general $F(x, y)$, the transformation (5) cannot be adopted, because $F(x, y)$ is not linear. However, the use of a generated map to construct a meta-map remains valid in a general $F(x, y)$ case, and a hierarchical configuration can exist for a particular configuration.

Appendix B. Multi-branch Nagumo–Sato map

In general, f_n with (at least) two type-I fixed points has the potential of possessing a Nagumo–Sato map as a generated map. To consider the general situation, we define the 'multi-branch Nagumo–Sato map' by (6), restricted within a region $I = [x_0, x_{n-1}]$, while $x_0, x_1, \ldots, x_{n-1}$ can be arranged arbitrarily. This type of map can be generated from random initial conditions.

In this map, we can choose a function which determines a map producing cycle of any length of period. To illustrate this property, we study the case with some special configurations.

First, two type-I fixed points are assumed to be 0 and 1. For the sake of symmetry, we choose $\epsilon = \frac{1}{2}$. Then, the two branches are given by

$$g[0](x) = \tfrac{1}{2}x, \quad x \in I_0^2, \quad f(I_0^2) = 0, \qquad g[1](x) = \tfrac{1}{2}x + \tfrac{1}{2}, \quad x \in I_1^2, \quad f(I_1^2) = 1. \tag{B.1}$$

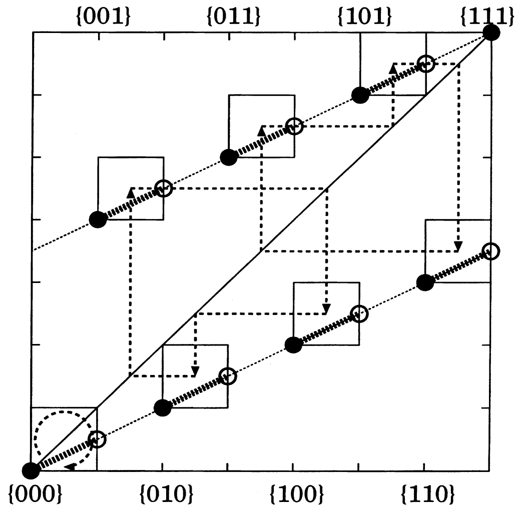

Fig. 13. A multi-branch Nagumo–Sato map $M_3(x)$. Two period-3 attractors coexist.

We represent a rational number a by the binary form $0.a_1a_2a_3\cdots$, with each $a_i = 0, 1$ (here $a = \sum_{k=1}^{\infty} a_k 2^{-k}$). In this representation, $g_0(x)$ acts as a right-shift, which acts as $0.a_1a_2a_3\cdots \to 0.0a_1a_2a_3\cdots$, and $g_1(x)$ acts as a right-shift and inserts 1 into the head of the sequence as $0.a_1a_2a_3\cdots \to 0.1a_1a_2a_3\cdots$. Hence these two branches act as 0, 1-inserter for a binary sequence.

Here we denote $a(a_i = a_{i+m})$ as $\{a_1a_2\cdots a_m\}$, while the set of m-length sequences $\{a_1a_2\cdots a_m\}$ is denoted by S_m. The number of elements which belong to S_m is 2^{m-1}, and the values of $a(\in S_m)$ take $i/2^{m-1}(i = 0, 1, \ldots, 2^{m-1})$. If $\{a_1a_2\cdots a_m\} \in S_m$, $\{a_m, a_1a_2\cdots a_{m-1}\} \in S_m$. Because of this, when $g(\{a_1a_2\cdots a_m\}) \equiv g_{a_m}(\{a_1a_2\cdots a_m\})$, the map $g(x)$ is a bijection $S_m \to S_m$. We define $M_m(x) = g(x)$ over S_m.

$M_m(x) \cup M_n(x)$ is a single-valued function for arbitrary m, n. The condition that a point $a \in S_m \cap S_n (m < n)$ exists is that m is a divisor of n. In such a case, a has the form $\{a_1a_2\cdots a_m\} \in S_m$, and $\{a_1a_2\cdots a_m\}^{n/m} \in S_n$. These two representations determine the same $g(a)$. Then, $M_\infty(x)$ defined as $\cup_{k=1}^{\infty} M_k(x)$ has an infinite period.

The function $M_m(x)$ is defined at 2^{m-1} points. With an appropriate arrangement, it is possible for $g(x)$ generated by the attractor of our functional dynamics to be made equal to $M_m(x)$ for all x. As an example, we define $g(x)$ as $g_{o(i)}$ for an interval $[i/2^{m-1}, i + 1/2^{m-1})$ $(i = 0, 1, \ldots, 2^{m-1} - 1)$. Here, $o(i) = 0$ for even i and $o(i) = 1$ for odd i. In Fig. 13, we can take a section$_i$ $[i/2^{m-1}, (i + 1)/2^{m-1}) \times [g(i/2^{m-1}), g(i/2^{m-1}) + 1/2^{m-1})$ within the region where $g(x)$ is defined. The map $g(x)$ determines the bijection section$_i$ $\to$ section$_j$. In each section, $g(x)$ has a slope $\frac{1}{2}$ (<1), and all orbits converge to attractors which are determined by $M_m(x)$. Thus we can embed a multi-branch Nagumo–Sato map which has multiple attractors.

In the same way, we can construct an n-branch Nagumo–Sato map. We assume $x_i = i/n, i = 0, 1, \ldots, n - 1$. If $\epsilon = (n - 1)/n$, each branch has the form

$$g[i](x) = \frac{1}{n}x + \frac{i}{n}\left(x \in I_i^2, f(I_i^2) = \frac{i}{n}\right). \tag{B.2}$$

Each branch $g[i](x)$ indicates a right-shift and insertion of i at the head of the n-digit sequence. Using these branches, we can embed an m-periodic point for an n-digit representation $\{a_0, a_1, \ldots, a_{m-1}\}(a_i = 0, 1, \ldots, n - 1)$.

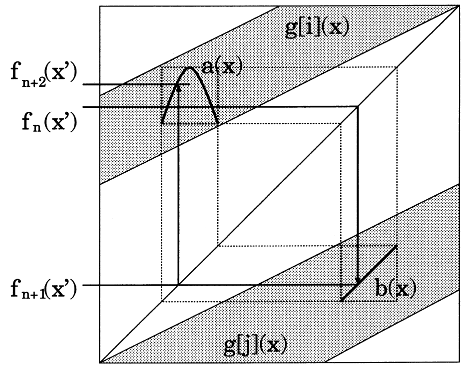

Fig. 14. An example with an arbitrary one-dimensional map embedded. Once the map is determined as in this figure, $f_n(x')$, which takes a value in this region, evolves according to $f_{2n+1}(x') = a(f_{2n}(x'))$ and $f_{2n+2}(x') = b(f_{2n+1}(x')) = b \circ a(f_{2n}(x'))$. If $b(x)$ is the identity function, $f_{2n+2}(x') = a(f_{2n}(x'))$. An arbitrary one-dimensional map can be embedded by observing the dynamics of the function every two steps.

Appendix C. Embedding a general one-dimensional map as a generated map

Let us examine closely the configuration of the type-I fixed intervals adopted to embed a one-dimensional map. The area in which a one-dimensional map is embedded has to be on the intersection between each branch of type-II intervals and $I \times \bar{I}_i^1$ (see Fig. 6). This implies that we cannot embed a map which is continuous around the identity function. However, one can embed an arbitrary one-dimensional map by considering a two-step iteration, i.e., as a map to generate $f_{n+2}(x)$ from $f_n(x)$. As shown in Fig. 14, let us take two maps in the dotted areas of Fig. 6. As shown in the figure, the generated maps $g[0](x)$ and $g[1](x)$ are put in two regular square sections. Here, $f_n(x')$ which is mapped to $g[0](x)$ evolves as $f_{n+1}(x') = g[0](f_n(x'))$ and $f_{n+2}(x') = g[1] \circ g[0](f_n(x'))$. If $g[0](x)$ is the identity function, the time evolution of $f_n(x')$ at $n = 2i$ (i is an integer) is $f_{n+2}(x') = g[1](f_n(x'))$, $f_{n+4}(x') = g[1](f_{n+2}(x'))f_{n+2i} = a^i(f_n(x'))$. Hence an arbitrary one-dimensional map can be embedded as a rule for the two-step iteration of the functional dynamics.

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