Available online at www.sciencedirect.com
SCIENCE DIRECT
Physica A 324 (2003) 372–379
PHYSICA A
www.elsevier.com/locate/physa

A non-linear model of economic production processes

A. Ponzi[a,*], A. Yasutomi[b], K. Kaneko[a]

[a]Department of Pure and Applied Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan
[b]Department of Area Studies, Graduate School of Arts and Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

Abstract

We present a new two phase model of economic production processes which is a non-linear dynamical version of von Neumann's neoclassical model of production, including a market price-setting phase as well as a production phase. The rate of an economic production process is observed, for the first time, to depend on the minimum of its input supplies. This creates highly non-linear supply and demand dynamics. By numerical simulation, production networks are shown to become unstable when the ratio of different products to total processes increases. This provides some insight into observed stability of competitive capitalist economies in comparison to monopolistic economies. Capitalist economies are also shown to have low unemployment.

PACS: 87.23.Ge; 87.23.Kg; 89.65.Gh; 89.75.Fb

Keywords: Economic dynamics; Evolutionary systems; Reaction networks; Non-linear dynamics

1. Introduction

In this paper, we present for the first time a new non-linear dynamical model of economic production, demand and supply processes which is based on von Neumann's neoclassical model of economic production [1,2]. We describe and motivate this model in some detail and then exhibit some results of its numerical simulation which have relevance for real economic dynamics.

The original von Neumann model (VNM) of balanced economic growth assumes that each good is produced jointly with certain others, in an analogous way to a chemical

* Corresponding author.
E-mail address: adam@complex.c.u-tokyo.ac.jp (A. Ponzi).

0378-4371/03/$ - see front matter
doi:10.1016/S0378-4371(02)01885-X

reaction. In this model capital goods at different stages of wear and tear are different goods and a production process is defined as operation which converts one bundle of goods, including capital equipment, into another bundle of goods, including the capital goods (at appropriate wear and tear) which are carried forward for use in further production. Capital goods therefore function something like *catalysts* in chemical reactions, reformed at the end of the reaction in an amount conserved in the reaction.

In the VNM it is assumed that consumption of goods takes place only through the processes of production, including necessities of life consumed by workers, and all income is reinvested in production. In the VNM there are a fixed number of processes and a fixed number of products in the system. Therefore the VNM is defined by a fixed input matrix and a fixed output matrix, representing, respectively, the fixed ratios of input products and output products for each process. Each process is assumed to have unit time duration, longer processes may be broken down into several processes with intermediate products each with unit time duration. Furthermore each process has an intensity, and each product has a price. For example a process may be given by: inputs: baker, dough, oven, coal and lunch; outputs: baker, oven, bread and waste (including heat). The ratios these products 'react' in are of course fixed and the baker and oven are catalysts (except for some 'wear and tear'). The intensity would correspond to the amount of bakers, etc., employed (the quantity of baker here is considered a continuous variable like the dough). An 'auto-catalytic' economic process may be for example: inputs: untrained workers, company training department; output: larger company training department.

The VNM is defined as a static equilibrium model describing relationships between the variables which must hold at equilibrium. Equilibrium is a state of 'balanced growth' where prices are constant and intensities of production grow or decay at constant geometric rates. There are no dynamics defined by the model which might describe out of equilibrium or approach to equilibrium behaviour. Indeed the VNM does not include a price setting market, where demand and supply are compared, which is necessary for a full dynamical model of production. The VNM does not account for many of the problematic, non-equilibrium behaviour of real economies, such as unstable prices, inflation and deflation, and unemployment for example. We have made a two-phase dynamical version of the VNM which includes a 'market phase' as well as a 'production phase' and while still only a 'toy' economy may shed some light on the above problems.

2. Model

The system has a fixed number of processes N denoted $i = 1,\ldots,N$ and a fixed number of products P denoted $j = 1,\ldots,P$. The product network is represented by an fixed input matrix a_{ij} representing the stoichiometric ratio of product j required by process i and a fixed output stoichiometric matrix b_{ij} representing the outputs of process i. In our simulations we fix $\sum_j a_{ij} = \sum_j b_{ij}$ for all processes i (although this is not physically necessary since heat may also be produced). The system has a production phase, where products are produced and a market phase, where they are valued and exchanged. This is the first difference between this model and the VNM, the next difference is that each process is considered to have its own 'stocks' at each time t, denoted $S_{ij}(t)$. Furthermore each process need not possess stocks $S_{ij}(t)$ in their exact stoichiometric ratios for the production reaction to proceed perfectly without any unused input. To take this into account we have to consider how the rate of the production reaction depends on the stocks. In a chemical reaction system this rate is governed by the law of mass action [3] which states that reactions proceed at rates proportional to the product of the concentrations of the input species. This law which depends on the consideration of random collisions as well as on the notion of volume is obviously inapplicable in an economic reaction context. In fact the rate determining quantity in an economic reaction is the quantity of the *minimum* input stock $S_{ij}(t)$ possessed by the process (i.e., company/country). For example a baker working at full pace can only fill a certain amount of ovens, increasing the amount of ovens further will not increase the rate of production of bread. Similarly employing more bakers will not increase your bread production if the ovens are already full. Therefore the process rate of production i, $R_i(t)$ is given by

$$R_i(t) = \alpha \, Min_j \left(\frac{S_{ij}(t)}{a_{ij}} \right) , \tag{1}$$

where $0 < \alpha < 1$ is a parameter, its meaning explained below. We consider this model to be a two phase mapping and the production phase is therefore defined by

$$S_{ij}\left(t + \tfrac{1}{2}\right) = S_{ij}(t) + (\beta b_{ij} - a_{ij}) R_i(t) , \tag{2}$$

where $0 < \beta < 1$ is a parameter expressing 'wear and tear' in the production process. All processes are considered to proceed simultaneously. We have written $S_{ij}\left(t + \frac{1}{2}\right)$ to denote the fact that this is only half of a two phase process.

In the VNM all processes also have an intensity. Instead of intensity in this model we consider each process to have a *value* called *funds* $F_i(t)$ measured in dollars. Funds are necessary for the trading process and are a generalized *fitness* measuring how successful the process has been in the past. Funds are not necessarily held by the process but may simply indicate the *ability to borrow* from banks which is of course directly dependent on the value of the company. In the same way as the VNM we consider all funds to be reinvested in production every time step, and that therefore the amount of funds controls the intensity of the process in the subsequent production phase. At the start of the trading phase therefore each process i trys to buy as much of each product as it can for its funds and constructs its *demands* $D_{ij}\left(t + \frac{1}{2}\right)$, which are measured in dollars, for products j, by appropriately dividing its available funds $F_i(t)$ $\left(= F_i\left(t + \frac{1}{2}\right)\right)$. However in the allocation of demand the process does not know in advance the price which will be collectively formed in the market. In reality this may be a complex iterated process. However the most natural and realistic 'rule of thumb' for a process to allocate its funds to demand may be the following

$$D_{ij}\left(t + \tfrac{1}{2}\right) = F_i(t) \frac{a_{ij} p_j(t)}{\sum_j a_{ij} p_j(t)} , \tag{3}$$

where $p_j(t)$ $\left(= p_j\left(t + \frac{1}{2}\right)\right)$ is the price of product j formed during the previous trading phase, i.e., processes divide funds into fractions given by the latest product prices and stoichiometric ratios a_{ij}. The actual price at which transactions occur $p(t+1)$ is then made by matching demand and supply so that

$$p_j(t+1)S_j\left(t+\tfrac{1}{2}\right)=D_j\left(t+\tfrac{1}{2}\right)\ , \tag{4}$$

where $S_j(t)=\sum_i S_{ij}(t)+S_j^{ext}(t)$ is the total supply of product j at time t and $D_j(t)=\sum_i D_{ij}(t)+D_j^{ext}(t)$ is the total demand for product j, measured in dollars, and $S_j^{ext}(t)$ and $D_j^{ext}(t)$ are possible external supplies and demands. Product and fund allocation is then carried out in a straight forward 'mean-field' way so that

$$F_i(t+1)=\sum_j S_{ij}\left(t+\tfrac{1}{2}\right)p_j(t+1)\ , \tag{5}$$

$$S_{ij}(t+1)=\frac{D_{ij}\left(t+\frac{1}{2}\right)}{p_j(t+1)}\ . \tag{6}$$

This means that the funds $F_i(t)$ used to make the demands appearing in Eq. (3) are simply the money made from sales in the previous trading phase. It is important to realize that in this trading process there may be no net exchange of goods. In fact, in the case of factors of production, which are both demanded and supplied by each process in each trading phase, the process sells them at price $p_j(t+1)$ and buys them at the *same* price $p_j(t+1)$. The trading process defined by Eqs. (5), (6) and (4) should therefore be considered a *revaluation* process. If for example the price $p_j(t+1)$ has fallen a small amount from the previous time $p_j(t)$, (i.e., overall supply has increased) while the process funds $F_i(t)$ have stayed constant, then the trading-revaluation phase amounts to the small expansion of production by increasing the supplies of factors of production according to Eq. (6).

Eq. (3) assumes that the best indicator of $p_j(t+1)$ is $p_j(t)$. This may not necessarily be the case, other prescriptions may be to base demands on $\langle p_j\rangle$ where $\langle\cdots\rangle$ denotes a time average over the past, or simply to divide the funds according to the stoichiometric ratios a_{ij} which would be best if the process assumes subsequent prices are uncorrelated random numbers. We therefore generalise Eq. (3) to

$$D_{ij}\left(t+\tfrac{1}{2}\right)=F_i(t)\left(\gamma\frac{a_{ij}p_j(t)}{\sum_j a_{ij}p_j(t)}+(1-\gamma)a_{ij}\right)\ , \tag{7}$$

where γ is a parameter measuring how much a process takes the previous price into account in determining its current demands.

The parameter α appearing in Eq. (1) measures the rate of production, it can therefore be seen as a ratio of the two timescales, production and marketing. When it is small products are revalued and exchanged on a faster time scale than production. As it stands the model is not completely sound since the price can diverge or be zero. To avoid this we assume that in the price setting trading phase demands and supplies cannot be exchanged smaller than a small quantity $\varepsilon\sim 10^{-8}$, i.e., for example if $D_{ij}(t)$ is evaluated to be smaller than ε then $D_{ij}(t)$ is set to zero and likewise for supplies in trading.

3. Results

In all simulations reported here the product network is the simplest possible: each process has one input, one output and one catalyst, as would describe a chemical reaction network in a living cell, see for example Ref. [4]. (In real systems this single catalyst will correspond to human labour of course.) Here each process is considered to employ equal proportions of its inputs, therefore each process has one product with $a_{inprod} = 0.5$ and $b_{inprod} = 0$, one product with $a_{catprod} = 0.5$ and $b_{catprod} = 0.5$ and one product with $a_{outprod} = 0$ and $b_{outprod} = 0.5$ while all the other products have a_j and b_j zero. The production network is fixed and defined randomly at the start of each numerical simulation such that each product has at least one process which produces it and at least one process which consumes it. The system considered here is closed i.e., $S_j^{ext}(t) = D_j^{ext}(t) = 0$ for all j and the parameters are $\alpha = 0.9$, $\beta = 1.0$ and $\gamma = 0.5$. There are N products and P processes where the $3P$ initial stocks $S_{ij}(0)$ and P initial funds $F_j(0)$ are chosen uniformly randomly and the initial prices are $p_j(0) = 1$. The total supply $S(t) = \sum_{ij} S_{ij}(t) = K_1$ and funds $F(t) = \sum_i F_i(t) = K_2$ are both conserved quantities in the closed system with $\beta = 1.0$.

This model shows the full spectrum of dynamical behaviour including fixed and periodic points, heteroclinic cycles, chaos and intermittency the analysis of which is still in progress. A typical time series segment of the product supplies $S_j(t) = \sum_i S_{ij}(t)$ is shown in Fig. 1. The time series is very complex exhibiting multi-timescale cyclic behaviour often observed in real economies. This timescale complexity is created by a complex switching process whereby the rates $R_i(t)$ of the production processes are

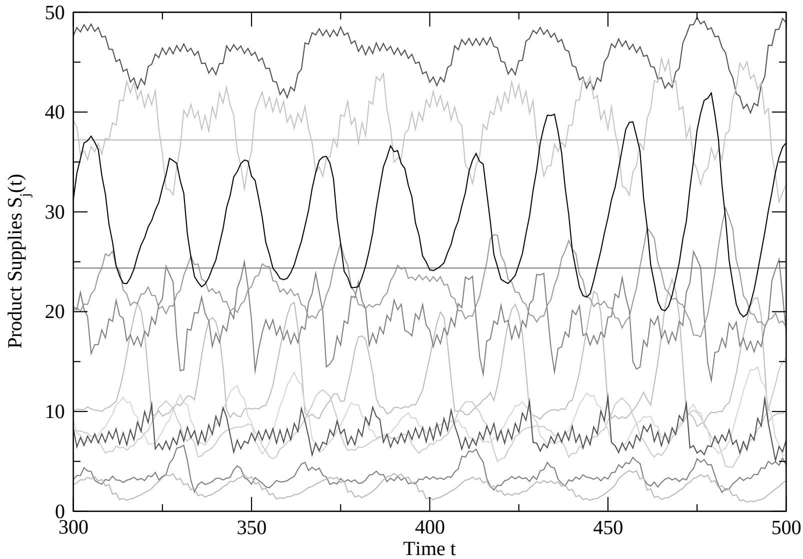

Fig. 1. Time series of product supplies $S_j(t)$ for $N = 30$ processes, $P = 20$ products system. (Not all products are plotted.)

alternately controlled by the different supply rates of the different input products according to Eq. (1). Notice there are some products at fixed supply. In fact this can occur because the trading process conserves $S_j(t)$, although the production process does not, of course. As explained initially the network is defined such that each product has at least one demand process and one supply process. Therefore this means some processes have 'died' or 'gone out of business', due to competition from other processes. By 'died' we mean that its funds $F_i(t)$ are given by $F_i(t) < \varepsilon$, and therefore it makes no demands, i.e., $D_{ij}(t) < \varepsilon$, and furthermore this immediately means its supplies $S_{ij}(t)$ are also $S_{ij}(t) < \varepsilon$ according to Eq. (6). (The parameter ε here is only a convenience to deal with finite numerical precision, indeed this process death occurs no matter how small we set ε). Therefore the conserved products appearing in the time-series are simply factors of production in the remaining 'live' processes. This would correspond to a country which does not produce some necessary factor of production such as cars for example, in fact if we had chosen $\beta < 1.0$ in this case the network would not survive, without external supply of energy. In fact which processes die and which do not is a complex problem dependent on the network structure as well as on the other parameters. An important parameter in this respect has turned out to the ratio of products to processes P/N.

Shown in Fig. 2(a) are ensemble average results as the amount of products P is varied for fixed amount of processes $N=30$ and in Fig. 2(b) as the amount of processes N is varied for fixed amount of products $P = 10$. The results shown are time averages over 3×10^5 iterations after discarding a transient of 3×10^6 iterations. Furthermore each point is an ensemble average of 30 trial time-series with different networks and initial conditions. The diamonds show the proportion of processes N^{live}/N which had non-zero funds for the entire time series, i.e., $F_i(t) > \varepsilon$. This result shows that as P/N increases the network can support fewer and fewer processes. At first sight this seems strange since there are more products so one may think more processes can stay in business. However few processes producing a product cause large fluctuations in supply and price due to the highly non-linear dynamics created by the minimum condition in Eq. (1) interacting with the marketing dynamics. In fact the dynamics is much more likely to be chaotic (in preparation) for large P/N while the small P/N dynamics is usually characterized by fixed points and (quasi-)periodic behaviour. This can be immediately seen as a vindication of *capitalism* and the *stable* economy it is observed to create where many processes compete to produce a product. Indeed large P/N is characteristic of a monopolistic economy.

Furthermore the crosses show the mean production i.e., $\langle R(t)\rangle$ where total production $R(t) = \sum_i R_i(t)$ is given according to Eq. (1), and $\langle \cdots \rangle$ here denotes averaging over the process production time series and over all 30 trials. We have divided by the theoretical maximum production of $K_1/2N$. This shows a decay with increase in P/N. Since the total product supply K_1 is a conserved quantity, independent of the amount of processes alive N^{live}, this is a highly non-trivial result. In fact it clearly indicates increasing *unemployment* with P/N, since unemployment is given by $K_1 - \langle R(t)\rangle$. Of course in the large P/N monopolistic economy the unstable dynamics creates large differences in the supply of the input products, (and therefore large fluctuations in the prices), which means one them, the input or the catalyst, is left unused. This

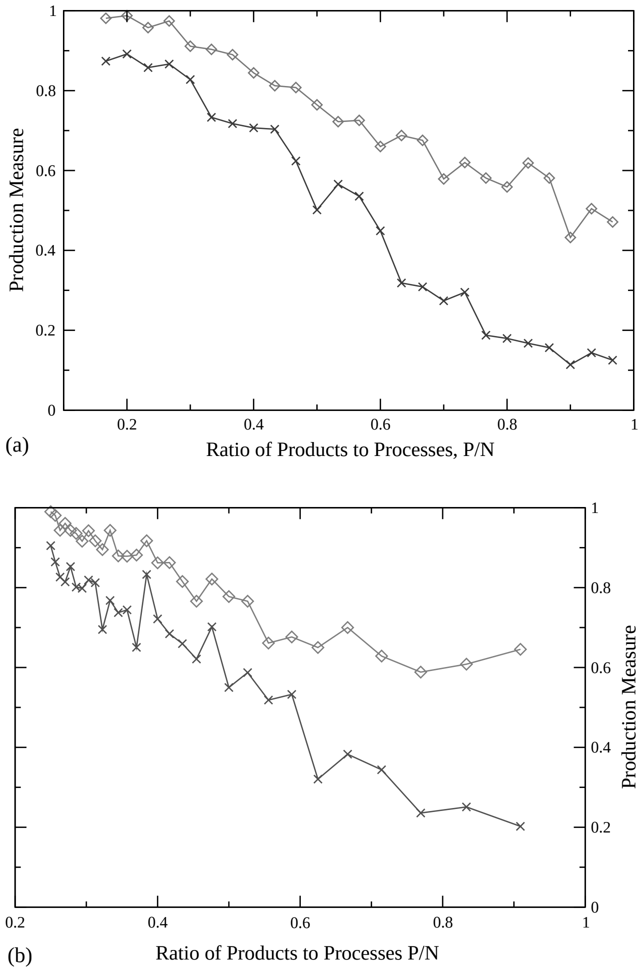

Fig. 2. ($\diamond$) Proportion of non-bankrupt processes N^{live}/N with $F_i(t) > \varepsilon$ and ($\times$) total productivity $\langle R(t)\rangle$. (a) No. of processes $N = 30$ fixed, no. of products P varied and (b) no. of products $P = 10$ fixed, no. of processes N varied.

quantity $\langle R(t)\rangle$ is also the best measure of the *efficiency* of the economy, since when $2NR(t)/K_1 = 1$ the processes are running perfectly, i.e., perfectly anticipating the future price.

4. Conclusion

We have presented a new model of two-phase production and marketing economy based on von Neumann's neoclassical model. The model exhibits complex realistic dynamics based on production rates determined by minimum supplies. We have shown that economic networks with fixed number of processes can only support a certain amount of products before becoming unstable as the number of products is increased. This situation is characterised by some processes 'going bankrupt' and the remaining processes operating inefficiently. This causes unemployment to appear. Work is still in progress to completely classify the dynamics of this system as its various parameters are varied, including open systems exchanging products with the environment (other countries), and systems generating waste and heat.

Acknowledgements

A. Ponzi wishes to thank the Japan Society for the Promotion of Science for support of this research.

References

[1] J. von Neumann, Rev. Econ. Stud. 31 (1945) 1.
[2] M. Morishima, Theory of Economic Growth, Oxford University Press, Oxford, 1970.
[3] P.W. Atkins, Physical Chemistry, Oxford University Press, Oxford, 1994.
[4] K. Kaneko, T. Yomo, Bull. Math. Biol. 59 (1997) 139.