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The systematic adiabatic elimination method is formulated using the eigenfunction expansion method, in terms of the small parameter γ^{-1}, which is the time scale for a fast variable. This method is applied to the linear processes with some examples of an optical system and a chemical reaction model. We also apply this method to the colored noise problems, to obtain the lowest correction of the effect of the memory.

§ 1. Introduction

Recently many people are interested in multiplicative stochastic processes[3)~10)] and colored noise processes[11)~13)] as fascinating problems of nonequilibrium statistical physics. The origin of multiplicative stochastic processes is thought to be due to adiabatic elimination of fast variables[6),14)] and to fluctuations of the surroundings.[3)~10)] However, the elimination of fast variables in stochastic processes is not so clear as in deterministic processes and has rather delicate problems such as the problems of Itô or Stratonovich stochastic calculus.[14),15)] No systematic method with the use of a small parameter has been developed so far. In § 2, we present a systematic method of elimination of fast variables, using the eigenfunction expansions. The time scale τ for the fast variables is used as a small parameter. Applications to linear processes are given in § 3, with an example of a simplified model of Raman scattering. The threshold of the laser action may be reduced by the fluctuation of the external pumping. Other applications to linear processes are presented in § 4 with a simple example to a chemical reaction model. There we calculate the higher order correction of the small parameter τ. We note that our theory corresponds to the Born-Oppenheimer approximation in quantum mechanics, where coordinates of electrons are fast variables.

Colored noise processes are also interesting problems in nonequilibrium statistical physics. Only two-level colored noises are treated rigorously,[11),12)] and for colored Gaussian noises we have to resort to the expansion method using small parameters such as τ, the correlation time of the colored noise. This problem is related to the adiabatic elimination, since we can consider that colored noises appear as a result of the elimination of fast variables. In § 5, we apply the eigenfunction expansion method to this problem and obtain a Fokker-Planck type equation to order of τ. Discussions are given in § 6.

In this paper we will mainly describe our formulation. Some examples of the applications of this method to more realistic systems, such as the optical problems, will be reported in a separate paper, with some extensions of this paper.

§ 2. General formulation of the adiabatic elimination

We present in this section a systematic method of elimination of fast variables in stochastic processes. For simplicity, we confine ourselves only to a two-variable system (one slow variable and one fast variable which we eliminate).

We consider the following Langevin equations:

$$\begin{cases} \dot{x} = f(x, y) + \xi_x(t), \\ \dot{y} = -\gamma a(x, y) + \sqrt{\gamma}\,\xi_y(t), \end{cases} \tag{2·1}$$

where $\xi_x(t)$ and $\xi_y(t)$ are Gaussian white noises satisfying

$$\langle \xi_x(t)\xi_x(t')\rangle = Q_x \delta(t-t'),$$

$$\langle \xi_y(t)\xi_y(t')\rangle = Q_y \delta(t-t')$$

and

$$\langle \xi_x(t)\xi_y(t')\rangle = 0 . \tag{2·2}$$

Here, γ is a large parameter and we use $1/\gamma$ as a small parameter, in order to eliminate the fast variable y. We assume that the quantities $f(x, y)$ and $a(x, y)$ are $O(\gamma^0)$. The factor $\sqrt{\gamma}$ in front of $\xi_y(t)$ in Eq. (2·1) appears, since the stationary distribution of the variable y is postulated to be independent of γ when x is fixed.

The Fokker-Planck equation associated with Eq. (2·1) is

$$\frac{1}{\gamma}\frac{\partial}{\partial t}P(x, y; t) = -\frac{1}{\gamma}\frac{\partial}{\partial x}f(x, y)P + \frac{Q_x}{2\gamma}\frac{\partial^2}{\partial x^2}P$$
$$+\frac{\partial}{\partial y}a(x, y)P + \frac{Q_y}{2}\frac{\partial^2}{\partial y^2}P . \tag{2·3}$$

In order to project $P(x, y, t)$ into $P(x, t) = \int P(x, y, t)dy$, we first expand

$$P(x, y; t) = \sum_n P_n(x, t)\Pi_n(y; x), \tag{2·4}$$

where $\{\Pi_n(y; x)\}$ are eigenfunctions of the Fokker-Planck equation

$$\frac{Q_y}{2}\frac{\partial^2}{\partial y^2}\Pi_n(y; x) + \frac{\partial}{\partial y}a(x, y)\Pi_n(y; x) = -\lambda_n(x)\Pi_n(y; x). \tag{2·5}$$

This is a Fokker-Planck equation for the variable y driven by the force $-a(x, y)$ with x as a parameter. Here, we note that $\lambda_0(x)=0$ (the eigenvalue for the stationary solution) and $\lambda_n(x)>0$ for $n\neq 0$. Using the well-known transformation into the Schrödinger-type equation from the Fokker-Planck equation,[16] we have

$$\Pi_n(y; x)=\phi_n(y; x)\phi_0(y; x), \tag{2·6}$$

where $\{\phi_n(y; x)\}$ are eigenfunctions of the following 'Schrödinger equation',

$$Q_y\lambda_n(x)\phi_n(y; x)=\left[-\frac{Q_y^2}{2}\frac{\partial^2}{\partial y^2}+V(y; x)\right]\phi_n(y; x) \tag{2·7}$$

and

$$V(y; x)=\frac{1}{2}a(x, y)^2+\frac{Q_y}{2}\frac{\partial a(x, y)}{\partial y}. \tag{2·8}$$

We note that $\{\phi_n(y; x)\}$ satisfy the orthonormality condition $\int\phi_n(y; x)\phi_m(y; x)dy=\delta_{nm}$. The projection of the probability distribution into the x-space is performed as follows:

$$P(x, t)\equiv\int P(x, y, t)dy=P_0(x, t), \tag{2·9}$$

because $\int\Pi_n(y; x)dy$ vanishes except for $n=0$.

Inserting Eq. (2·4) into Eq. (2·3) and integrating over y after multiplying by ϕ_n/ϕ_0, we obtain

$$\begin{aligned}\frac{1}{\gamma}\frac{\partial}{\partial t}P_n(x, t)=&-\frac{1}{\gamma}\sum_k\langle\phi_k(y; x)f(x, y)\phi_n(y; x)\rangle\frac{\partial P_k}{\partial x}\\&-\frac{1}{\gamma}\sum_k P_k\left\langle\frac{\phi_n}{\phi_0}\frac{\partial}{\partial x}(f\phi_k\phi_0)\right\rangle\\&+\frac{1}{2\gamma}Q_x\left\{\frac{\partial^2}{\partial x^2}P_n+2\sum_k\frac{\partial P_k}{\partial x}\left\langle\frac{\phi_n}{\phi_0}\frac{\partial}{\partial x}(\phi_k\phi_0)\right\rangle\right.\\&\left.+\sum_k P_k\left\langle\frac{\phi_n}{\phi_0}\frac{\partial^2}{\partial x^2}(\phi_k\phi_0)\right\rangle\right\}-\lambda_n(x)P_n(x, t),\end{aligned} \tag{2·10}$$

where $\langle\cdots\rangle$ denotes $\int\cdots dy$.

Then the equation for $P_0(x, t)$ takes the form

$$\begin{aligned}\frac{\partial}{\partial t}P_0(x, t)=&-\frac{\partial}{\partial x}\langle f(x, y)\phi_0^2(y; x)\rangle P_0(x, t)+\frac{Q_x}{2}\frac{\partial^2}{\partial x^2}P_0(x, t)\\&-\sum_{k\neq 0}\frac{\partial}{\partial x}\langle\phi_k(y; x)f(x, y)\phi_0(y; x)\rangle P_k(x, t),\end{aligned} \tag{2·11}$$

since

$$\frac{\partial}{\partial x}\langle\phi_0(y;x)\phi_k(y;x)\rangle=\frac{\partial}{\partial x}\delta_{k0}=0\,.$$

In order to obtain the closed equation for $P_0(x,t)$ to $O(\gamma^{-1})$, we need the lowest contribution of $P_k(k\neq 0)$ to P_0. This contribution is given by

$$\lambda_k(x)P_k(x,t)=\frac{1}{\gamma}\Big\{-\langle\phi_k f\phi_0\rangle\frac{\partial}{\partial x}-\Big\langle\frac{\phi_k}{\phi_0}\frac{\partial(f\phi_0{}^2)}{\partial x}\Big\rangle$$
$$+Q_x\Big(\Big\langle\frac{\phi_k}{\phi_0}\frac{\partial\phi_0{}^2}{\partial x}\Big\rangle\frac{\partial}{\partial x}+\frac{1}{2}\Big\langle\frac{\phi_k}{\phi_0}\frac{\partial^2\phi_0{}^2}{\partial x^2}\Big\rangle\Big)\Big\}P_0(x,t)+O\Big(\frac{1}{\gamma^2}\Big).$$

(2·12)

Thus, we obtain

$$\frac{\partial}{\partial t}P(x,t)+\frac{\partial}{\partial x}\langle f(x,y)\phi_0{}^2(y;x)\rangle P(x,t)-\frac{Q_x}{2}\frac{\partial^2}{\partial x^2}P(x,t)$$
$$=\frac{1}{\gamma}\sum_{k\neq 0}\frac{\partial}{\partial x}\langle\phi_k(y;x)f(x,y)\phi_0(y;x)\rangle\frac{1}{\lambda_k(x)}$$
$$\times\Big\{\frac{\partial}{\partial x}\langle\phi_k(y;x)f(x,y)\phi_0(y;x)\rangle$$
$$+\Big\langle f(x,y)\phi_0{}^2(y;x)\Big(\frac{\partial}{\partial x}\frac{\phi_k(y;x)}{\phi_0(y;x)}\Big)\Big\rangle-2Q_x\Big\langle\phi_k\frac{\partial\phi_0}{\partial x}\Big\rangle\frac{\partial}{\partial x}$$
$$-\frac{Q_x}{2}\Big\langle\frac{\phi_k}{\phi_0}\frac{\partial^2\phi_0{}^2}{\partial x^2}\Big\rangle\Big\}P(x,t)+O\Big(\frac{1}{\gamma^2}\Big).\qquad(2\cdot 13)$$

This is a Fokker-Planck type equation for $P(x,t)$ up to order of γ^{-1}. Here, we note that, in general, a $(\partial/\partial x)^3$ term appears in the next order.

Adiabatic elimination in a deterministic equation, which is widely used,[1),2)] is easily reproduced from Eq. (2·13) as follows: As Q_y goes to 0,

$$\langle f(x,y)\phi_0{}^2(y;x)\rangle=\int f(x;y)P_{st}(y;x)dy\to f(x,y_0(x)),\qquad(2\cdot 14)$$

because $P_{st}(y;x)\to\delta(y-y_0(x))$ as $Q_y\to 0$. Here $y_0(x)$ is a stable solution of the equation $a(x,y)=0$. Thus, we obtain to the lowest order of γ^{-1},

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}f(x,y_0(x))P(x,t)\qquad(2\cdot 15)$$

if $Q_x=0$. That is, we obtain $\dot{x}=f(x,y_0(x))$.

When the noise strength Q_y is small, we can use WKB approximation for the estimation of $\phi_n(y;x)$. Only the small region near $y\approx y_0(x)$ contributes to the brackets in Eq. (2·13) in this case. We expand $a(x, y)$ around the most stable solution $y=y_0(x)$

$$a(x, y)=\left.\frac{\partial a}{\partial y}\right)_{y=y_0(x)}(y-y_0(x))+\cdots \tag{2·16}$$

and neglect the terms of $o(y-y_0)$. Then we can compute $\phi_k(y; x)$ and $\lambda_k(y; x)$ easily. This will be performed in the next section.

When the cross correlation $\langle\xi_x(t)\xi_y(t')\rangle=Q_{xy}\delta(t-t')$ does not vanish in Eq. (2·2), there appears the extra term $(Q_{xy}/\sqrt{\gamma})(\partial^2/\partial x\partial y)\ P(x, y; t)$ in Eq. (2·3). This gives no correction to Eq. (2·11), but gives the additional term

$$\frac{2Q_{xy}}{\sqrt{\gamma}}\left\{\frac{\partial}{\partial x}\left\langle\phi_k\frac{\partial\phi_0}{\partial y}\right\rangle-\left\langle\phi_0\frac{\partial\phi_0}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\phi_k}{\phi_0}\right)\right)\right\rangle\right\}P_0(x, t) \tag{2·17}$$

to Eq. (2·12). Thus we have to add to the right-hand side of Eq. (2·13) the following term:

$$-\frac{2Q_{xy}}{\sqrt{\gamma}}\sum_{k\neq0}\frac{\partial}{\partial x}\langle\phi_k f\phi_0\rangle\frac{1}{\lambda_k(x)}$$
$$\times\left\{\frac{\partial}{\partial x}\left\langle\phi_k\frac{\partial\phi_0}{\partial y}\right\rangle-\left\langle\phi_0\frac{\partial\phi_0}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\phi_k}{\phi_0}\right)\right)\right\rangle\right\}P(x, t). \tag{2·18}$$

When we take the corrections up to order of Q_{xy}^2/γ into account, there appears a $\partial^3/\partial x^3$ term and this is no longer of the Fokker-Planck type. In many cases, however, Q_{xy} is small and $O(1/\sqrt{\gamma})$, and we can neglect these corrections. We also note that we do not need the last two terms in Eq. (2·13) if Q_x is small and $o(\gamma^0)$.

§ 3. Applications to a linear process I

In this section we apply the method described in § 2 to some linear processes. Here, linear processes mean that the equation for the fast variable y is linear about y as will be seen in Eqs. (3·1) and (3·9). We consider the following Langevin equations:

$$\begin{cases}\dot{x}=f(x)+g(x)y+\xi_x(t),\\ \dot{y}=-\gamma c(x)\{y-a(x)\}+\sqrt{\gamma}\,\xi_y(t),\end{cases} \tag{3·1}$$

where $\xi_x(t)$ and $\xi_y(t)$ are Gaussian white noises with the time correlations obeying Eq. (2·2). The associated Fokker-Planck equation is

$$\frac{1}{\gamma}\frac{\partial}{\partial t}P(x, y, t)=-\frac{1}{\gamma}\frac{\partial}{\partial x}f(x)P$$
$$+\frac{Q_x}{2\gamma}\frac{\partial^2}{\partial x^2}P-\frac{1}{\gamma}y\frac{\partial}{\partial x}g(x)P$$

$$+\frac{\partial}{\partial y}(c(x)(y-a(x)))P+\frac{Q_y}{2}\frac{\partial^2}{\partial y^2}P\,. \tag{3·2}$$

Eigenfunctions $\Pi_n(y;x)$ which satisfy the equation

$$\frac{Q_y}{2}\frac{\partial^2}{\partial y^2}\Pi_n+\frac{\partial}{\partial y}(c(x)(y-a(x)))\Pi_n=-\lambda_n\Pi_n(y;x)$$
$$=-nc(x)\Pi_n(y;x) \tag{3·3}$$

are expressed by

$$\Pi_n(y;x)=\phi_n(y;x)\phi_0(y;x);$$
$$\phi_n(y;x)=\left(\frac{c(x)}{\pi Q_y}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,e^{-c(x)(y-a(x))^2/2Q_y}H_n\left(\sqrt{\frac{c}{Q_y}}(y-a)\right), \tag{3·4}$$

where $H_n(z)$ is the n-th Hermite polynomial.

Using the recursion formula for $H_n(z)$

$$H_n'(z)=2nH_{n-1}(z)\quad\text{and}\quad 2zH_n(z)=H_{n+1}(z)+2nH_{n-1}(z) \tag{3·5}$$

and the orthonormality of $\{\phi_n(z)\}$, we can compute easily the integrals appearing in Eq. (2·13) to obtain

$$\langle\phi_k(y;x)y\phi_0(y;x)\rangle=a(x)\delta_{k0}+\sqrt{\frac{Q_y}{2c(x)}}\delta_{k1} \tag{3·6}$$

etc. Thus we obtain from Eq. (2·13),

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}f(x)-\frac{Q_x}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}a(x)g(x)\right)P(x,t)$$
$$=\frac{1}{\gamma}\frac{\partial}{\partial x}\left(\frac{g(x)f(x)a'(x)+g(x)^2a(x)a'(x)}{c(x)}\right)P(x,t)$$
$$+\frac{Q_y}{2\gamma}\frac{\partial}{\partial x}\frac{g(x)}{c(x)}\frac{\partial}{\partial x}\frac{g(x)}{c(x)}P(x,t)$$
$$-\frac{Q_x}{\gamma}\frac{\partial}{\partial x}g(x)\left(\frac{a'(x)}{c(x)}\frac{\partial}{\partial x}+\frac{a''(x)}{2c}\right)P(x,t)+O\left(\frac{1}{\gamma^2}\right). \tag{3·7}$$

Here, we note that the conventional technique of the adiabatic elimination[7] gives only the second term on the right-hand side of Eq. (3·7).

When the cross correlation Q_{xy} does not vanish, we have to add to the right-hand side of Eq. (3·7) the following expression:

$$\frac{Q_{xy}}{\sqrt{\gamma}} \frac{\partial}{\partial x} \frac{g(x)}{c(x)} \frac{\partial}{\partial x} P(x, t) \tag{3·8}$$

from Eq. (2·18).

When Q_x is $O(1/\gamma)$ and Q_{xy} is $O(1/\sqrt{\gamma})$, Eq. (3·7) without Q_x/γ terms and Eq. (3·8) yield the Fokker-Planck equation for $P(x, t)$ up to order of $1/\gamma$. This agrees with the result obtained by Morita, Mori and Mashiyama.[14]

Next, we consider the following case:

$$\begin{cases} \dot{x} = f(x) + g(x)y + h(x)y^2 + \dfrac{1}{\sqrt{\gamma}}\xi_x(t), \\ \dot{y} = -\gamma c(x)(y - a(x)) + \sqrt{\gamma}\,\xi_y(t). \end{cases} \tag{3·9}$$

Calculations are straightforward and we obtain the Fokker-Planck equation for the distribution of x up to order of γ^{-1}:

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} f - \frac{Q_x}{2\gamma} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} ag + \frac{\partial}{\partial x}\left(a^2 + \frac{Q_y}{2c}\right)h\right] P(x, t) \\ &= \frac{1}{\gamma} \frac{\partial}{\partial x}(g + 2ah)\left[\frac{a'f}{c} + \frac{gaa'}{c} + \frac{ha'}{c}\left(a^2 + \frac{Q_y}{2c}\right) + \frac{Q_y}{2c} \frac{\partial}{\partial x} \frac{g + 2ah}{c}\right] P(x, t) \\ &\quad + \frac{Q_y}{\gamma} \frac{\partial}{\partial x}\left[-\frac{c'}{4c^3} hf + \frac{g}{2\sqrt{c}}\left(\frac{a}{\sqrt{c}}\right)' + \frac{Q_y}{4} \frac{h}{\sqrt{c}} \frac{\partial}{\partial x} \frac{h}{c^{5/2}} + \frac{h^2}{2c^{3/2}}\left(\frac{a^2}{\sqrt{c}}\right)'\right] P(x, t) \\ &\quad + \frac{Q_{xy}}{\gamma}\left(\frac{\partial}{\partial x} \frac{g + 2ah}{c} \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \frac{a'h}{c}\right) P(x, t). \end{aligned} \tag{3·10}$$

Here, we note that the drift part is given by

$$-\frac{\partial}{\partial x}\left[f + ag + \left(a^2 + \frac{Q_y}{2c}\right)h\right] P(x, t) \tag{3·11}$$

to the lowest order of γ^{-1} and the diffusion part is given by

$$\frac{1}{\gamma} \frac{\partial^2}{\partial x^2}\left[\frac{Q_x}{2} + (g + 2ah)^2 \frac{Q_y}{2c^2} + \frac{Q_y{}^2 h^2}{4c^2} + \frac{Q_{xy}}{c}(g + 2ah)\right] P(x, t) \tag{3·12}$$

to the lowest order.

As an example, we consider a simplified model of Raman scattering.[6] Considering four-wave mixing processes, we can write the following field equations:

$$\begin{cases} \dot{\tilde{A}}_S{}^+ = -\gamma_S \tilde{A}_S{}^+ + \chi|A_L|^2 \tilde{A}_S{}^+ + \xi_S{}^+, \\ \dot{A}_L{}^+ = -\gamma_L A_L{}^+ - \chi|\tilde{A}_S|^2 A_L{}^+ + \tilde{P}^+ + \sqrt{\gamma_L}\,\xi_L{}^+, \end{cases} \tag{3·13}$$

where $\tilde{A}_S^{\pm}$, $A_L^{\pm}$ and $\tilde{P}^{+}$ represent the Stokes-scattered light amplitude, the laser mode, and the external source respectively. The Gaussian white noises are represented by ξ_S^{+} and ξ_L^{+}. As a simplified model, we neglect the effect of the phase and consider the quantities appearing in Eq. (3·13) to be real. Putting A_S as $\tilde{A}_S/\sqrt{\gamma_L}$ and P as $\tilde{P}/\gamma_L$, we obtain

$$\begin{cases} \dot{A}_S = -\gamma_S A_S + x A_L^2 A_S + \dfrac{1}{\sqrt{\gamma_L}}\xi_S\,, \\ \dot{A}_L = -\gamma_L\{(1+xA_S^2)A_L - P\} + \sqrt{\gamma_L}\xi_L\,. \end{cases} \tag{3·14}$$

When γ_L is large compared with γ_S, we can eliminate the laser field adiabatically. Using the formula (3·10), we obtain the Fokker-Planck equation for A_S to order of γ_L^{-1}. Corresponding to Eqs. (3·10) and (3·11), the drift term and the diffusion term are given by

$$-\frac{\partial}{\partial A_S}\left\{-\gamma_S A_S + \left(\frac{P^2}{(1+xA_S^2)^2} + \frac{Q_L}{2(1+xA_S^2)}\right)xA_S\right\}P(A_S, t) \tag{3·15}$$

and

$$\frac{1}{\gamma_L}\frac{\partial^2}{\partial A_S^2}\left\{\frac{Q_S}{2} + \frac{2P^2x^2A_S^2}{(1+xA_S^2)^4}Q_L + \frac{Q_L^2x^2A_S^2/4 + 2Q_{SL}PxA_S}{(1+xA_S^2)^2}\right\}P(A_S, t) \tag{3·16}$$

respectively. Here Q_S denotes the strength of the random force ξ_S and Q_L denotes that of ξ_L, and Q_{SL} is the strength of the cross correlation.

The stationary distribution $P_{st}(A_S)$ is easily obtained to the lowest order of γ_L^{-1} from Eqs. (3·15) and (3·16),

$$P_{st}(A_S) \propto \exp\left[\gamma_L \int^{A_S} dx \frac{x\left\{-\gamma_S + x\left(\dfrac{P^2}{(1+xx^2)^2} + \dfrac{Q_L}{2(1+xx^2)}\right)\right\}}{\dfrac{Q_S}{2} + \dfrac{2P^2x^2x^2}{(1+xx^2)^4}Q_L + \dfrac{Q_L^2x^2x^2/4 + 2Q_{SL}Pxx}{(1+xx^2)^2}}\right]. \tag{3·17}$$

Thus the threshold is given by $P_{thr} = \sqrt{\gamma_S/x - Q_L/2}$. Here, the threshold is defined by the value of the pumping, beyond which the distribution function $P_{st}(A_S)$ has its maximum at $A_S \neq 0$. We note that the fluctuation Q_L reduces the threshold. We may observe this effect by fluctuating the strength of the external pumping.

§ 4. Applications to a linear process II

In this section we consider the following Langevin equations:

$$\begin{cases} \dot{x} = f(x) + g(x)y + \xi_x(t), \\ \dot{y} = -\gamma c(x)y + a(x) + \sqrt{\gamma}\,\xi_y(t), \end{cases} \tag{4·1}$$

where $\xi_x(t)$ and $\xi_y(t)$ are Gaussian white noises satisfying Eq. (2·2). We expand $P(x, y; t)$ by the Hermite polynomials as in § 3,

$$P(x, y; t) = \sum_n \sqrt{c(x)}\,P_n(x, t)\,Y_n(\sqrt{c(x)}\,y), \tag{4·2}$$

where

$$Y_n(z) = \left(\frac{z}{\pi Q_y}\right)^{1/2} \frac{1}{\sqrt{2^n n!}} e^{-z^2} H_n\left(\frac{z}{\sqrt{Q_y}}\right) H_0\left(\frac{z}{\sqrt{Q_y}}\right). \tag{4·3}$$

Using the recursion relations (3·5), we have

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} f - \frac{Q_x}{2}\frac{\partial^2}{\partial x^2}\right) P_0 = -\sqrt{\frac{Q_y}{2}}\frac{\partial}{\partial x}\left(\frac{g}{\sqrt{c(x)}} P_1(x)\right) \tag{4·4}$$

and we obtain the contribution of $P_1(x)$ to $P_0(x)$ in order of γ^{-2}, when Q_x is $O(1/\gamma)$

$$P_1(x) = -\frac{1}{\gamma}\left(1 - \frac{1}{\gamma c}\left(\frac{\partial}{\partial t} + \sqrt{c}\frac{\partial}{\partial x}\frac{f}{\sqrt{c}}\right)\right) \times \left(\sqrt{\frac{Q_y}{2c}}\frac{\partial}{\partial x}\frac{g}{c} - \frac{a}{c}\sqrt{\frac{2c}{Q_y}}\right) P_0 - \frac{1}{2\gamma^2}\sqrt{\frac{Q_y}{2c}}\frac{\partial}{\partial x}\frac{gc'}{c^3} f P_0 . \tag{4·5}$$

Inserting Eq. (4·5) into Eq. (4·4), there appear terms $(1/\gamma^2)(\partial/\partial t)(\partial/\partial x)$ and $(1/\gamma^2)(\partial/\partial t)(\partial^2/\partial x^2)$ which are not of the Fokker-Planck type. However, these terms can be eliminated in order of γ^{-2}, by multiplying Eq. (4·4) by $\{(1+(1/\gamma^2) \times ((Q+2)(\partial/\partial x)(g/c^2)(\partial/\partial x)(g/c) - (\partial/\partial x)(ag/c^2))\}^{-1}$ from the left. Thus we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} f - \frac{Q_x}{2}\frac{\partial^2}{\partial x^2} - \frac{Q_y}{2\gamma}\frac{\partial}{\partial x}\frac{g}{c}\frac{\partial}{\partial x}\frac{g}{c} + \frac{1}{\gamma}\frac{\partial}{\partial x}\frac{a}{c}\right) P(x, t)$$

$$= \frac{Q_y}{2\gamma^2}\frac{\partial}{\partial x}\frac{g}{c}\left\{\frac{1}{c}\frac{\partial}{\partial x}\frac{1}{c}(gf' - fg')\right.$$

$$+\frac{5}{2c}\frac{\partial}{\partial x}\frac{fgc'}{c^2}+\frac{1}{2}\frac{fgc'^2}{c^4}-\frac{g(f'c)'}{c^3}\Big\}P(x,t)$$

$$+\frac{1}{\gamma^2}\frac{\partial}{\partial x}\frac{fg}{c}\left(\frac{a}{c}\right)'P(x,t). \tag{4·6}$$

Here we note that in the case $Q_x=O(\gamma^0)$, a $(\partial/\partial x)^3$ term appears. When we assume $c(x)=1$, for example, we have up to order of γ^{-2}

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}f-\frac{Q_x}{2}\frac{\partial^2}{\partial x^2}\right)P(x,t)-\left(\frac{Q_y}{2\gamma}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g-\frac{1}{\gamma}\frac{\partial}{\partial x}ag\right)P(x,t)$$

$$=\frac{Q_y}{2\gamma^2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}(gf'-fg')P(x,t)+\frac{Q_xQ_y}{4\gamma^2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}\left(g''+2g'\frac{\partial}{\partial x}\right)P(x,t)$$

$$+\frac{1}{\gamma^2}\frac{\partial}{\partial x}(ga'f)P(x,t)-\frac{Q_x}{2\gamma^2}\frac{\partial}{\partial x}g\left(a''+2a'\frac{\partial}{\partial x}\right)P(x,t). \tag{4·7}$$

Before considering a simple example, we remark that when $y=p$, $f(x)=0$, $g(x)=1$, $c(x)=1$ and $Q_x=0$, Eq. (4·1) reduces to the Ornstein-Uhlenbeck process and the problem is the derivation of Smoluchowski equation with some corrections. This problem was treated by many physicists.[15),18)~22)] The eigenfunction expansion method was performed in this special case by Stratonovich at first[18)] and has been refined up to the present.[19),20)] We also note that the correction of $O(\gamma^{-2})$ vanishes in this case and the equation for $P(x,t)$ is of the Fokker-Planck type up to order of γ^{-3}.

As an example of Eq. (4·7), we consider the following chemical reaction model

$$\begin{cases} A+X\rightleftarrows mX\,, \\ Y+C\rightleftarrows X\,, \\ Y\rightarrow D\,. \end{cases} \tag{4·8}$$

The Langevin equations for this model are

$$\begin{cases} \dot{x}=ax-bx^m+cy+\xi_x(t), \\ \dot{y}=-\gamma y+kx+\sqrt{\gamma}\,\xi_y(t). \end{cases} \tag{4·9}$$

Assuming that the damping coefficient γ is large, we can use Eq. (4·7) to obtain the Fokker-Planck type equation for $P(x,t)$ in order of γ^{-2}. Here, we only write the expression of the stationary distribution function $P_{st}(x)$ as

$$P_{st}(x)$$

$$\sim\exp\left[\frac{\left(a+\frac{kc}{\gamma}-\frac{akc}{\gamma^2}\right)x^2-2\left(1-\frac{kc}{\gamma^2}-\frac{Q_yc^2ma}{\gamma^2Q_x}\right)\frac{bx^{m+1}}{m+1}-\frac{mbQ_yc^2}{\gamma^2}x^{m-1}-\frac{Q_yc^2x^{2m}}{\gamma^2Q_x}}{\left(Q_x+\frac{Q_xc^2}{\gamma}+\frac{Q_yc}{\gamma^2}(ac-2k)\right)}\right]. \tag{4·10}$$

From this expression, we can see the shift of the phase transition point[7] up to order of γ^{-2}, which is $a=0$ in the absence of the coupling between X and Y.

§ 5. An application to the colored noise system

Recently Sancho and San Miguel studied the following stochastic process:[13]

$$\dot{x}=f(x)+g(x)\eta(t), \tag{5·1}$$

where $\eta(t)$ is a Gaussian noise which satisfies

$$\langle\eta(t)\eta(t')\rangle=\frac{Q\gamma}{2}e^{-\gamma|t-t'|}. \tag{5·2}$$

Instead of Eqs. (5·1) and (5·2), we consider the equivalent coupled Langevin equations

$$\begin{cases}\dot{x}=f(x)+g(x)\eta\,,\\ \dot{\eta}=-\gamma\eta+\gamma\xi(t),\end{cases} \tag{5·3}$$

where $\xi(t)$ is a Gaussian white noise obeying

$$\langle\xi(t)\xi(t')\rangle=\frac{Q}{2}\delta(t-t'). \tag{5·4}$$

Then the problem is reduced to the elimination of the fast variable η and we can use the method described in preceding sections.

After scaling η by $y=\gamma^{-1/2}\eta$ and expanding $P(x,y;t)$ by

$$P(x,y,t)=\sum_n P_n(x,t)\,Y_n(y), \tag{5·5}$$

where Y_n is defined by Eq. (4·3), we obtain from the recursion relations (3·5)

$$P_n=\sqrt{\frac{Q}{2\gamma}}G_n\{\sqrt{n}P_{n-1}+\sqrt{n+1}P_{n+1}\} \tag{5·6}$$

and

$$G_n = \left(n + \frac{1}{\gamma}\frac{\partial}{\partial t} + \frac{1}{\gamma}\frac{\partial}{\partial x}f\right)^{-1}\frac{\partial}{\partial x}g\,. \tag{5·7}$$

Using the above relations, we can calculate $P_0(x)$, which is equal to the distribution function $P(x,t) = \int P(x,y;t)dy$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f(x)\right)P(x,t) = \frac{Q}{2}\frac{\partial}{\partial x}g\left\{G_1P_0 + \frac{Q}{\gamma}G_1G_2G_1P_0 + \cdots\right\}. \tag{5·8}$$

From this expansion, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f(x)\right)P(x,t)\\ &= \frac{Q}{2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}gP - \frac{Q}{2\gamma}\frac{\partial}{\partial x}g\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f\right)\frac{\partial}{\partial x}gP + \frac{Q^2}{4\gamma}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\frac{\partial}{\partial x}gP\end{aligned} \tag{5·9}$$

up to order of γ^{-1}. Multiplying Eq. (5·9) by $(1 + (Q/2\gamma)(\partial/\partial x)g(\partial/\partial x)g)^{-1} \approx (1 - (Q/2\gamma)(\partial/\partial x)g(\partial/\partial x)g)$ from the left, we obtain the Fokker-Planck type equation up to order of γ^{-1}.

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}f(x)P + \frac{Q}{2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}\left(g - \frac{1}{\gamma}(g'f - f'g)\right)P + O\left(\frac{1}{\gamma^2}\right). \tag{5·10}$$

This agrees with the result of Ref. 13), which was obtained by a more complicated method.

Equation (5·8) can be written in a compact form using continued fraction expansions. In fact, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f(x)\right)P(x,t)\\ &= \gamma\mathcal{Y}_0 \cfrac{}{\mathcal{F}_1 - \mathcal{Y}_1 \cfrac{}{\ddots\ \mathcal{Y}_{n-1}\cfrac{}{\mathcal{F}_n - \mathcal{Y}_n \cfrac{}{\vdots}\mathcal{Y}_n}\mathcal{Y}_{n-1}\ \ddots}\mathcal{Y}_1}\mathcal{Y}_0P(x,t),\end{aligned} \tag{5·11}$$

where $\mathcal{X}_n$ and $\mathcal{Y}_n$ are the operators defined by

$$\mathcal{X}_n = n + \frac{1}{\gamma}\frac{\partial}{\partial t} + \frac{1}{\gamma}\frac{\partial}{\partial x}f(x) \tag{5·12}$$

and

$$\mathcal{Y}_n = \sqrt{n+1}\sqrt{\frac{Q}{2\gamma}}\frac{\partial}{\partial x}g\,. \tag{5·13}$$

However, these corrections higher than $O(\gamma^{-1})$ are rather complex and not of the Fokker-Planck type.

§ 6. Discussion

In this paper, we have formulated the adiabatic elimination by the eigenfunction expansion method. We have confined our arguments in this paper only to the process with one fast variable and one slow variable. Extensions of our formulation to the process with many degrees of freedom will be reported in a separate paper.

The method described in § 2 is very general and will be applied to various problems of stochastic processes. Especially, optical problems are very interesting for the application of this method. We point out the possibility that the fluctuation may reduce the threshold of the laser action in Raman scattering in § 3.

We note that the adiabatic elimination, derivation of the Smoluchowski equation with corrections, and colored noise processes, which have been studied separately, have very common aspects and can be treated in a unified manner, as we have described in this paper.

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