Periodic lattices of chaotic defects

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A type of lattice in which chaotic defects are arranged periodically is reported for a coupled map model of open flow. We find that temporally chaotic defects are followed by spatial relaxation to an almost periodic state, when suddenly another defect appears. The distance between successive defects is found to be generally predetermined and diverging logarithmically when approaching a certain critical point. The phenomena are analyzed and shown to be explicable as the results of a boundary crisis for the spatially extended system.

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For their extremely rich phenomenology and their far reaching significance in the study of universal properties in complex systems, coupled map lattices (CML's) have drawn much interest in recent years. Among the most widely studied CML's are lattices of coupled logistic maps since these are of fundamental importance for gaining and deepening the understanding of spatially extended dynamical systems [1–8].

In this paper, we investigate the dynamics of a one-way coupled logistic lattice (OCLL), which is conceptually closely related to open flow systems and therefore relevant for the general study of turbulence (e.g., in technological applications like jet engines), pipe flow, and data traffic. In such systems, performance may critically depend on nonlinear effects and consequently it is of great importance to obtain insights into the underlying universal mechanisms. Indeed, we believe that some of our findings may experimentally be verified, as possibly in Ohtsuka and Ikeda's optical system with distributed nonlinear elements [9] or pipe flow [10].

In a previous paper [11], we reported the discovery of spatial chaos with temporal periodicity and analyzed the stability of the spatial patterns with a spatial map which enabled us to predict certain types of nontrivial downflow behavior. In principle, our analysis was restricted to large coupling strengths though. In the present paper, we will investigate small coupling strengths which in contrast to larger ones yield extremely rich temporal dynamics. We report the discovery of a fascinating type of lattice in which evenly spaced chaotic defects form a periodic pattern. It is shown that this phenomenon can be completely explained by considering a low-dimensional map, and that it is related to a boundary crisis.

The model under investigation can be expressed as

$$x^i_{n+1} = (1-\epsilon)f(x^i_n) + \epsilon f(x^{i-1}_n), \tag{1}$$

where n is the discrete time, i the discrete space, ϵ the coupling constant, and f the logistic map which has the nonlinearity α as its parameter [4]. We choose a fixed boundary condition set to 1.0 throughout, but the exact value does not influence the results in a qualitative way.

A rough phase diagram indicating the main patterns relevant to this paper is given in Fig. 1. The lower gray area stretching from $\alpha = 1.6$ to $\alpha = 2.0$ is the so-called zigzag region which mostly, except for the lower boundary, has a spatial and temporal periodicity of 2. The two temporal phases are furthermore spatially exactly out of phase and consequently there are only two fixed points x^*_1 and x^*_2 that can easily be found as

$$x^*_{1,2} = \frac{1 \pm \sqrt{4(1-2\epsilon)^2\alpha + 4\epsilon - 3}}{2(1-2\epsilon)\alpha}. \tag{2}$$

For $\alpha \gtrsim 1.76$ we observed the occurrence of coherent zigzag structures within the chaotic sea which are like islands that appear, disappear, grow, and shrink. Hence the corresponding area is marked as "Zigzag Islands."

Our main discovery here is that for $\alpha \lesssim 1.76$, above the zigzag regime, lattices emerge with evenly spaced defects whose internal dynamics is chaotic when starting from random initial conditions. An example is shown in Fig. 2, where the distance between successive defects is 20 lattice sites.

We would like to stress here that this periodicity is in

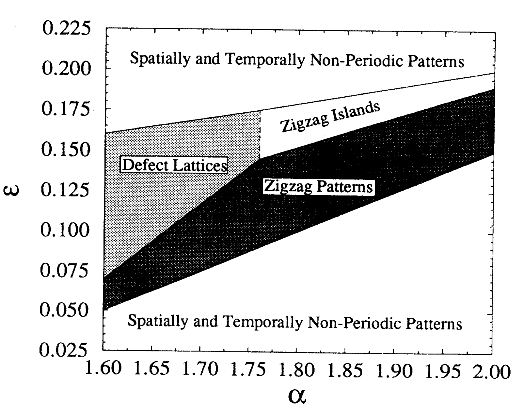

FIG. 1. Rough phase diagram for Eq. (1). The lines are only intended for indicating the approximate region in which the patterns occur. The exact boundaries may depend on both the boundary and initial conditions.

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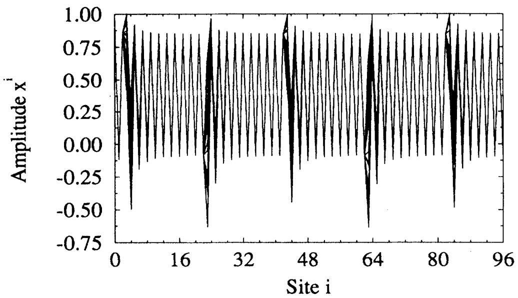

FIG. 2. Defect lattice in which the distance between defects is 20 lattice sites. The nonlinearity is $\alpha = 1.7$ and the coupling strength is $\epsilon = 0.115\ 95$.

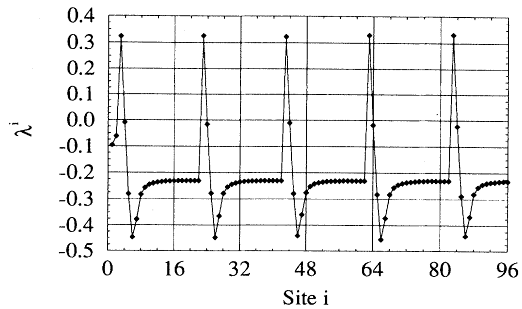

FIG. 4. Stationary Lyapunov exponents corresponding to Fig. 2. The product of 10 000 Jacobi matrices was taken. For each defect, there is only one lattice site with a positive Lyapunov exponent. The nonlinearity is $\alpha = 1.7$ and the coupling strength is $\epsilon = 0.115\ 95$.

the spatial direction, and that the periodicity can be very long. As is shown in Fig. 3, we find that the distance between defects diverges logarithmically when approaching a critical value $\epsilon_c \approx 0.115\,25$ beyond which the system is attracted to the zigzag solution for all initial conditions. The distance between defects throughout a lattice is nearly constant and increases in basic steps of 1 when approaching ϵ_c (from a minimum of three sites). Since there are two phases, this yields lattices in which either most of the successive defects are in the same phase, or the phases alternate.

In order to obtain an impression of the extent to which the chaotic motion of the defects is localized, the stationary Lyapunov exponents corresponding to Fig. 2, which can immediately be found as the eigenvalues of the product of Jacobi matrices [4]

$$\lambda^i = \ln(1-\epsilon) + \frac{1}{T}\sum_{n=1}^{n=T} \ln f'(x_n^i), \qquad (3)$$

are depicted in Fig. 4. Due to the upper triangle of the Jacobi matrix being zero for Eq. (1), there is no mixing and the ith exponent represents the local chaoticity of the ith site.

As can readily be seen from Figs. 2 and 4, the chaotic motion is generally localized at one lattice site, while it is damped in the down-flow direction. The surprising aspect here is that the last site before a new defect shows neither any significant remnant motion, nor a substantial deviation from the zigzag solution which is known to be absolutely stable. That is to say, it is not only stable in the stationary frame, but also in any comoving frame [6,11].

This is also clearly visible in the spatial return map [2,3] given in Fig. 5, where the straight lines indicate that there is only a very weak correlation between the value of a site prior to the defect and the defective site itself. In other words, the zigzag solution is stable to at least several percent of global and thus also local noise (here, by global noise we indicate the case that noise is added at every lattice site, and by local noise the case that noise is only added to one specific site), but in the defect lattice we first see damping to far within the basin of the stable zigzag pattern and then suddenly another defect at a certain (minimum) distance.

In order to obtain a quantitative estimate on how close to the zigzag pattern the lattice is, we plotted $x^{2i} - x^*$ versus space in Fig. 6 with i shifted such that the defect is at site $i = 1$. The average of 1000 samples per every second time step was taken of the sites following the

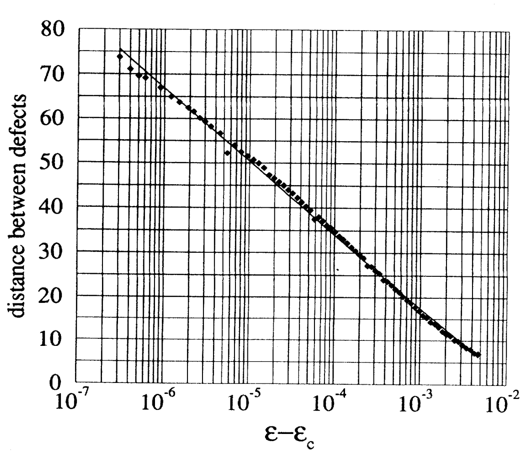

FIG. 3. Distance between successive defects versus $\epsilon - \epsilon_c$ with $\epsilon_c = 0.115\ 25$. The nonlinearity is $\alpha = 1.7$, and 2.5×10^6 time steps were discarded for each point calculated.

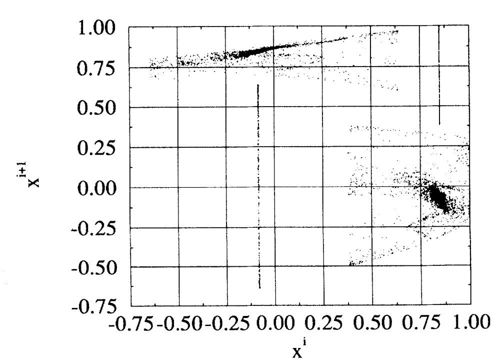

FIG. 5. Spatial return map of Fig. 2 with 50 second iterates overlaid. The straight lines clearly indicate that the lattice sites preceding chaotic ones all have (nearly) the same constant amplitudes. The nonlinearity is $\alpha = 1.7$ and the coupling strength is $\epsilon = 0.115\ 95$.

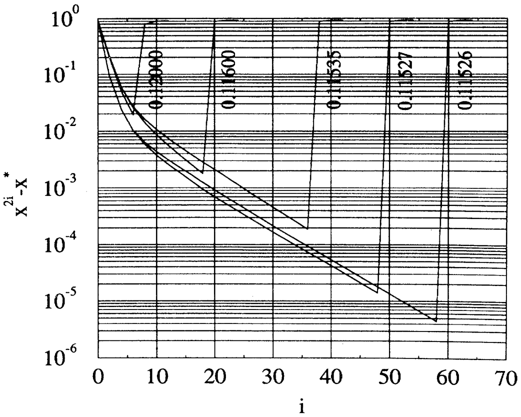

FIG. 6. Distance between the defect lattice and the zigzag pattern. The numbers indicate the values of the coupling constant ϵ, and the nonlinearity is $\alpha = 1.7$.

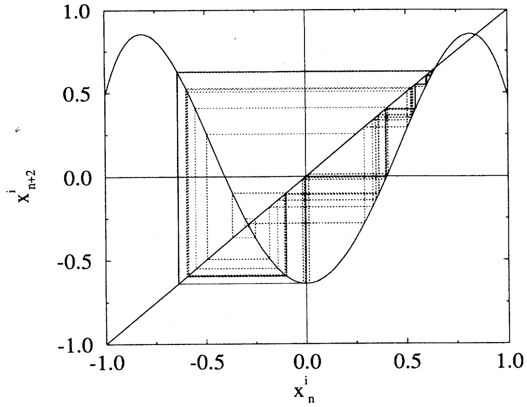

FIG. 7. Plot of Eq. (4) before the critical point $\epsilon_c \approx 0.115\,25$. The nonlinearity is $\alpha = 1.7$ and the coupling constant is $\epsilon = 0.12$. The dotted line indicates the first 50 iterates of 0.0.

first defect after a spatial transient of 100 sites. What is notable is that, when approaching the critical value ϵ_c, we do not see a decay in the damping rate as is the case for the usual critical phenomena. Instead we see that a perturbation remains decaying exponentially to x^* regardless of the proximity to ϵ_c. Consequently, there must be some threshold related to the phenomenon itself which is lowered.

The notion of (initially) counterintuitive behavior is reinforced further by our finding that the addition of a sufficiently large (but not too large) amount of global noise drives the entire lattice to the zigzag attractor instead of increasing the irregular dynamics, while a small amount of noise leads to a decrease in the distance between defects (i.e., an increase in the density of defects). We furthermore wish to stress that Fig. 2 is not a "lucky" special case, but the general final state for the given parameters when starting from random initial conditions.

Hence we have several apparently opposing tendencies, i.e., we obtain localized chaos instead of chaotic motion countering regularity, sites well in the basin of the zigzag attractor are followed by a defect instead of a regular site, and, an increase of global noise yields more regularity instead of less regularity. In order to unravel these, we analyze the phenomena described above by employing the fact that the sites just before a defect virtually assume the values of the zigzag pattern. The second iterate of Eq. (1) can then be approximated as

$$x^i_{n+2} = (1-\epsilon)f((1-\epsilon)f(x^i_n) + \epsilon f(x^*_2)) + \epsilon f(x^*_1), \quad (4)$$

where x^*_1 and x^*_2 are the stable zigzag solutions.

This enables us to plot x^i_{n+2} as a function of x^i_n as in Figs. 7 and 8 which depict the situations just before and just after the critical point ϵ_c. From these figures we can infer that for $\epsilon > \epsilon_c$ there are two distinct basins: one for a chaotic attractor and one for the fixed point of the zigzag pattern, while for $\epsilon < \epsilon_c$ there is only one basin. At ϵ_c we therefore have a boundary crisis [12] at which the boundary of the chaotic attractor (not the boundary of the basin of the chaotic attractor), in this case given by the second iterate of the origin, touches an unstable fixed point.

We find that the pattern formation of the lattice can be associated with the following scenario. When starting from random initial conditions, some lattice sites will be attracted to the zigzag attractor. Let us assume x^{i-1} is one such site. The question arising now is under what circumstances will x^i be a defect? If $\epsilon < \epsilon_c$ we have only one basin of attraction, and (eventually) all defects will disappear. If $\epsilon > \epsilon_c$, however, we find two distinct basins, implying that a site in the chaotic basin is (at least initially) a defect.

Since the gap between the chaotic attractor and the basin boundary of the zigzag attractor is quite narrow for $\epsilon > \epsilon_c$, a small modulation of x^{i-1} will create leaks to the zigzag basin. If such a leak exists, x^i cannot be a defect and it will eventually be attracted to the zigzag solution.

The zigzag pattern is a periodic attractor with a negative Lyapunov exponent however. Accordingly, modulation will be damped in the down-flow direction until

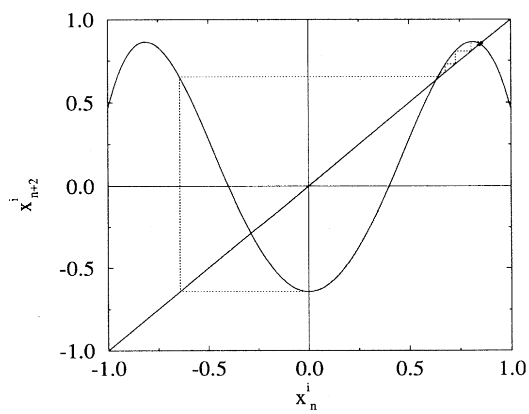

FIG. 8. Plot of Eq. (4) after the critical point $\epsilon_c \approx 0.115\,25$. The nonlinearity is $\alpha = 1.7$ and the coupling constant is $\epsilon = 0.11$. The dotted line indicates the first 25 iterates of 0.0.

at some site the modulation is too small to create leaks to the basin of the periodic attractor. If at this stage we have a site in the chaotic basin it will form a defect. When starting from random initial conditions, this will almost always be the case, and we obtain a periodic lattice.

A consequence of the presented scenario is that only the minimum distance between defects is determined. This can easily be verified by choosing a zigzag pattern as initial condition with defects inserted at selected sites. If the distance between two successive defects is larger than the minimum distance, nothing will happen, while if the distance between two successive defects is smaller than the minimally allowed distance, the down-flow defect will be pushed away. It should be mentioned here that it is not possible for a defect to be annihilated when located between (nearly) zigzag sites. This is due to the fact that a defect contains only one chaotic site and thus separates two zigzag regions with opposite phases (this is independent of the issue whether the sites preceding two defects are in the same phase since the distance can be even or uneven), and consequently, if the site of a defect leaks to the basin of the periodic attractor, automatically the next down-flow site ends up in the basin of the chaotic attractor.

Since the gap between the edge of the chaotic attractor and the unstable fixed point is approximately proportional to $\epsilon - \epsilon_c$, we have that the threshold t below which the chaotic motion must be damped (in order for the motion to be weak enough to allow the next site to be a defect) is also proportional to $\epsilon - \epsilon_c$ as can also immediately be inferred from Fig. 6. The Lyapunov exponent of the zigzag pattern is more or less constant close enough to ϵ_c, implying that $\delta^k \propto \exp(-k|\lambda|)$ with δ^k the size of a perturbation at a distance k from the perturbation. Hence the minimum distance k from a perturbation (and thus the minimum distance in the defect lattice) for which we have that $\delta^k < t$ is given by $k \propto \ln(\epsilon - \epsilon_c)$, yielding indeed the observed logarithmic divergence of the distances between defects.

In the present paper we have only studied the fully up-flow coupled case. We have confirmed however that the inclusion of a small down-flow coupling term (i.e., coupling to site x^{i+1}) does not affect our results qualitatively. We believe that this is a further indication that this mechanism of pattern formation may be universal and can be encountered elsewhere.

In the diffusively coupled logistic lattice, Brownian motion of defects has been reported [2] in a region of parameter space slightly above the regular zigzag regime. Although we could not observe any defect lattices, we nevertheless believe that the Brownian motion too is the result of a boundary crisis, the only difference with the one-way coupled case being that leaks always exist to some extent, allowing the defect to move to the right or left.

In conclusion, we found a type of lattice in which chaotic defects occur periodically when starting from random initial conditions. The encountered phenomena constitute a mechanism of pattern formation which was successfully analyzed and associated with a boundary crisis.

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