Sensitive boundary condition dependence of noise-sustained structure

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(Received 22 March 2000; revised manuscript received 6 September 2000; published 27 February 2001)

Sensitive boundary condition dependence (BCD) is reported in a convectively unstable system with noise, where the amplitude of generated oscillatory dynamics in the downstream depends sensitively on the boundary value. This BCD is explained in terms of the decrease (relaxation) of the comoving Lyapunov exponent (characterizing the convective instability) from upstream to downstream [K. Fujimoto and K. Kaneko, Physica D **129**, 203 (1999)]. It is shown that a fractal BCD appears if the dynamics that represent the spatial change of the fixed point includes transient chaotic dynamics, and if appropriate intensity of noise is presented. By considering as an example a one-way-coupled map lattice, this theory for BCD is demonstrated.

DOI: 10.1103/PhysRevE.63.036218 PACS number(s): 05.45.Ra, 05.40.Ca

I. INTRODUCTION

Study of dynamical system with spatial asymmetry is generally important in open fluid flow, chemical reaction with flow, electric circuit system, as well as in intracellular signaling system with successive autocatalytic reactions. In such system, downstream dynamics may depend sensitively on the *boundary* condition rather than *initial* condition. Such boundary condition dependence will be important generally in spatially extended dynamical systems, as well as in biological systems.

To consider this problem, convective instability plays an important role, since it causes amplification of a disturbance along flow [1–3]. If a system is convectively unstable (CU), a small disturbance at an upstream position is amplified and transmitted downstream. Due to this property, spatiotemporal structure with a large amplitude can be generated in the downstream by a tiny fluctuation in the upstream. Such structure is referred to as noise-sustained structure (NSS) [3].

Convective instability is quantitatively characterized by a co-moving Lyapunov exponent λ_v, i.e., the Lyapunov exponent observed in an inertial system moving with the velocity v [4,5]. If $\max_v \lambda_v$ is positive for a given state, the state is convectively unstable. This condition is compared to that for linear instability, implies $\lambda_0 > 0$. Absolute stability (AS), which implies stability along any flow [1,2], is guaranteed by the condition $\max_v \lambda_v < 0$. (With respect to an attractor, the co-moving Lyapunov exponent is used as an indicator of chaos: chaos with convective instability is characterized by the positivity of $\max_v \lambda_v$. However, with respect to a state, it is generally used to characterize its stability.)

In a system with convective instability and noise, we have shown that the downstream dynamics depend on the boundary condition at the upstream. Such behavior exhibiting some threshold-type dependence on the boundary condition was identified and analyzed in connection with the change of the convective instability along the flow [6]. In the present paper, we demonstrate that *sensitive* BCD of the downstream dynamics can appear in a class of noisy open-flow systems, and we clarify the condition for this appearance.

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In this paper we consider the simple case of a discrete system in one spatial dimension of the kind described above. As a simple example, we adopt a one-way coupled map lattice (OCML) [5–11] with noise:

$$x_n^i = (1-\epsilon) f(x_{n-1}^i) + \epsilon f(x_{n-1}^{i-1}) + \eta_n^i . \qquad (1)$$

Here n is a discrete time step, i is the index denoting elements ($i = 1,2, \ldots M =$ system size), and η is a white noise satisfying $\langle\langle \eta_n^i \eta_m^j \rangle\rangle = \sigma^2 \delta_{nm} \delta_{ij}$, with $\langle\langle \cdots \rangle\rangle$ representing an ensemble average ($\sigma \ll 1$) [12]. A *fixed boundary condition* x^0 is adopted and the (sensitive) dependence of the downstream dynamics on x^0 is studied. The use of a CML here is just for convenience for illustration. The results and theory we present are straightforwardly adapted to the case of coupled ordinary differential equations (ODE).

II. MECHANISM OF BOUNDARY DEPENDENCE

In the present case we choose the logistic map $f(x) = 1 - ax^2$ ($-1 < x < 1$). The parameters a and ϵ are chosen so that in the noiseless case all elements are attracted to fixed points $x_n^i = x_*^i$ for any initial and boundary conditions [i.e., the fixed points (in time) are AS in the downstream]. This attraction to fixed points is realized in the strong coupling regime (see Ref. [9]). The values of the fixed points can depend on the lattice site number (i.e., can be functions of space), and set of points $\{x_*^i\}$ form a spatially periodic pattern, as show in Fig. 1. When noise is added, however, it can be amplified in the downstream to create oscillating motion (see Fig. 2) if the upstream fixed points are convectively unstable [3].

Whether or not this noise-sustained structure is formed in the downstream depends on the boundary value x_0 and the noise strength. As a rough measure for the amplitude of the downstream oscillation, the root mean square (RMS) $\sqrt{\langle x_n^i - \langle x_n^i \rangle \rangle^2}$ is computed, where $\langle \cdots \rangle$ is the temporal average. As shown in Fig. 3, the RMS has a threshold-type dependence on x_0. Such a BCD has been observed in a one-way coupled ODE [6]. The mechanism responsible for the sensitive BCD clarified in that study is universal and can be summarized as follows: A change in x^0 causes a change in the value of upstream fixed points x_*^i, which, in turn, causes

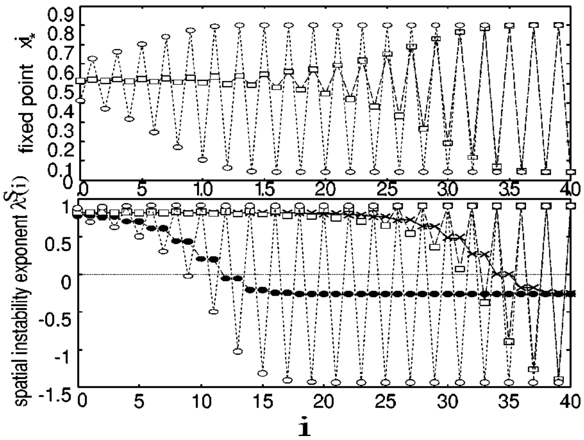

FIG. 1. The patterns x^i_* and $\lambda^S(i)$. The upper figure displays the fixed-point pattern x^i_* in the absence of noise, while the lower figure displays the spatial pattern of $\lambda^S(i)$. Two sets of plots are overlaid for different values of the boundary, $x^0=0.52$ (solid line with □) and 0.60 (dotted line with ○). For the lower figure $\Sigma_{\ell=0}^{1}\lambda^S(i-\ell)/2$ (× for $x^0=0.52$, ● for 0.60) is also plotted. This difference reflects the difference in the downstream dynamics in the case with noise. For the boundary value for $x^0=0.60$ no oscillation is generated in the downstream (since the CU region is narrow, as discussed in the text), while oscillation is generated in the downstream for $x^0=0.52$, as shown in Fig. 3. Here, $a=1.8$, $\epsilon=0.83$.

a change in the degree of convective instability. Accordingly the downstream dynamics, generated through the spatial amplification of noise, can be different for different values of x^0.

In the present case this mechanism is quantitatively analyzed as follows: The spatial fixed points dynamics are described by the spatial recursive equation [9]

$$x^i_* = (1-\epsilon)f(x^i_*) + \epsilon f(x^{i-1}_*) = \frac{-1+\sqrt{1+4a(1-\epsilon)(1-a\epsilon(x^{i-1}_*)^2)}}{2a(1-\epsilon)} \equiv g(x^{i-1}_*), \tag{2}$$

while the co-moving Lyapunov exponent $\lambda_v(i)$ of each fixed point x^i_* [6,10] is given by

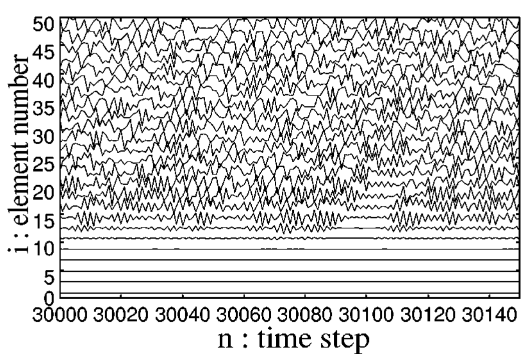

FIG. 2. Spatiotemporal plot of NSS with $\sigma=2\times10^{-3}$, $a=1.8$, $\epsilon=0.83$, and $x^0=0.52$. Here x^i_n is plotted for every second lattice site.

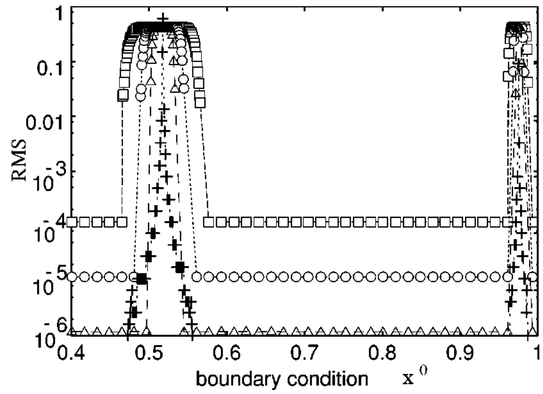

FIG. 3. BCD of the RMS and i_u-i_g (+, $\sigma=2\times10^{-4}$) for the downstream dynamics. NSS appears around $x^0=0.52$ and 0.97. Except these two regions for the boundary value, x^i_n in the downstream falls on fixed points. Different lines correspond to different noise intensities: $\sigma=2\times10^{-6}$(□), 2×10^{-5}(○), 2×10^{-4}(△). With the increase in the strength of the noise, the size of the region in which NSS is found increases. In the region $i_u-i_g>0$ (for $\sigma=2\times10^{-4}$), there exists noise-sustained structure. Here $\epsilon=0.842$, and the fixed point pattern is spatially period 2.

$$\lambda_v(i) = (1-\epsilon)\log|f'(x^i_*)| + \epsilon\log|f'(x^{i-1}_*)| + \log\left|\frac{1-\epsilon}{1-v}\right| + \log\left|\frac{\epsilon(1-v)}{v(1-\epsilon)}\right|^v. \tag{3}$$

A relevant quantity, characterizing the amplification of a disturbance per lattice site is given by the spatial instability exponent $\lambda^S(i)$ [13],

$$\lambda^S(i) = \max_v \frac{\lambda_v(i)}{v}. \tag{4}$$

In Fig. 1, $\lambda^S(i)$ is also plotted as a function of the lattice site number. For both the boundary values $x^0=0.52$ (□) and $x^0=0.60$ (○), convective instability exists in the upstream, as indicated by $\lambda^S(i)$. (Here the downstream pattern has spatial period 2, and $\lambda^S(i)$ also oscillates with this period. For this reason, $\Sigma_{\ell=0}{}^{1}\lambda^S(i-\ell)/2$, the average over the spatial period, is also plotted.) For $x^0=0.60$, $\lambda^S(i)$ (or the average over the spatial period) becomes negative at a smaller lattice site number than the case for $x^0=0.52$. In fact, as discussed below, this difference in the convergence rate of $\lambda^S(i)$ is relevant to the BCD of the downstream dynamics.

Since we have assumed that the fixed point pattern is AS in the downstream, the noise has to be amplified at a lattice point where the fixed point remains CU, in order for NSS to be formed. First, we estimate the lattice point i_u, defined as the site where the convective instability is lost; in other words, the site where the fixed point x^i_* changes from CU to AS. Recall that the x^i_* approach a periodic pattern in the downstream in the absence of noise. Denoting this spatial period by L, the lattice point i_u is given by the point i such that $\Sigma_{\ell=0}^{L-1}\lambda^S(i-\ell)/L$ is positive for $i<i_u$ and negative for $i\geqslant i_u$. As can be also expected by considering Figs. 1 and 3,

i_u is strongly correlated with the relaxation scale of the spatial map. If the convergence of the spatial map to its attractor is more rapid, i_u is generally smaller. (See Fig. 1: for example, $i_u = 12$ for $x^0 = 0.60$ indicated by ●, while $i_u = 36$ for $x^0 = 0.52$ indicated by $\times$.)

On the other hand, the scale i_g required for the amplification of a tiny noise to $O(1)$ can be estimated by

$$\sigma \exp\left(\sum_{i=1}^{i_g} \lambda^S(i) \right) \sim 1. \tag{5}$$

Then, the condition [6] for the formation of NSS is simply given by

$$i_u - i_g \geqslant 0. \tag{6}$$

In Fig. 3, we have plotted $i_u - i_g$ and the RMS of downstream dynamics as function of the boundary value x^0. The numerical results clearly support the conclusion that the condition for NSS is given by $i_u - i_g \geqslant 0$. Although this condition concerns only the sign of $i_u - i_g$, the amplitude of the NSS is also highly correlated with the value of $i_u - i_g$, as shown in Fig. 3. In the case of Fig. 3, the downstream AS dynamics are of spatial period 2, where NSS appears around $x^0 = 0.52$ and 0.97. The BCD here is simple, with just two regions allowing for NSS.

III. SENSITIVE BOUNDARY CONDITION DEPENDENCE

For the case of spatial period 4, as shown in Fig. 4(a), there are many undulations, and this BCD has a self-similar fine structure, to be shown as fractals [see the blow-up of Fig. 4(a)]. The complexity of this self-similar structure of the BCD increases with the period, as shown in Fig. 4(b) for the case of spatial period 16. With this self-similar structure, a small difference in the boundary value results in a large difference in the downstream dynamics.

Note that the loss of convective instability in the downstream (for the noiseless case) is necessary to have such BCD. If the downstream without noise is CU, then the BCD becomes weaker as the distance from the boundary increases, and in an infinitely large system, this BCD eventually dies away completely. To check the convective instability of the downstream, we have computed Lyapunov exponent of the spatial map in Eq. (2) [8] and the spatial instability exponent λ^S of the fixed points averaged over 16 lattice points at the downstream. In Fig. 5, we are plotted as a function of coupling strength ϵ. Since the absolute stability of the downstream is necessary to have BCD, it exists only at separated intervals in the parameter space [14].

In Fig. 6, dependence of RMS on the noise strength and boundary condition is plotted. The region in which the NSS with a large value of RMS form a tongue-like structures. For a small σ, say, less than $\sigma = 10^{-12}$ NSS is not formed for any boundary condition. In $\sigma = 10^{-3}$ some tongues merge as Fig. 6(b) and the complexity of the BCD decreases. Note that the complexity of the boundary condition dependence, measured by the number of ups and downs in the boundary condition dependence, takes its maximal value at the medium level of noise intensity, as shown in Fig. 7. It takes its maximal value at the medium level of noise intensity. If the intensity of the noise is too small, NSS is difficult to be formed, while if it is too large, the parameter domain to give each NSS is broadened as shown in Fig. 3 and Fig. 6, and the neighboring domains are fused successively with the increase of noise intensity, leading to the decrease in the number of undulation in the BCD. Accordingly, the BCD is most complex at some finite noise intensity, where the number of undulation in the BCD, shown in Fig. 4, of downstream amplitude takes a maximal value. The noise intensity that gives a peak of complexity in Fig. 7 depends on ϵ. This is explained in terms of the ϵ dependence of the convective instability $\lambda^S(<0)$ at the downstream, shown in Fig. 5. As λ^S increases toward 0, the (downstream) region with convective instability is longer. Hence, i_u (accordingly $i_u - i_g$) gets larger. As shown in Eqs. (5) and (6), the increase of $i_u - i_g$ gives the same influence on the dynamics as the increase of σ. Hence the noise intensity that gives the most complex BCD, is shifted to a smaller value, as shown by the shift from $\epsilon = 0.9108$ (solid line with $\times$) to $\epsilon = 0.928198$ (dotted line with ○) in Fig. 7.

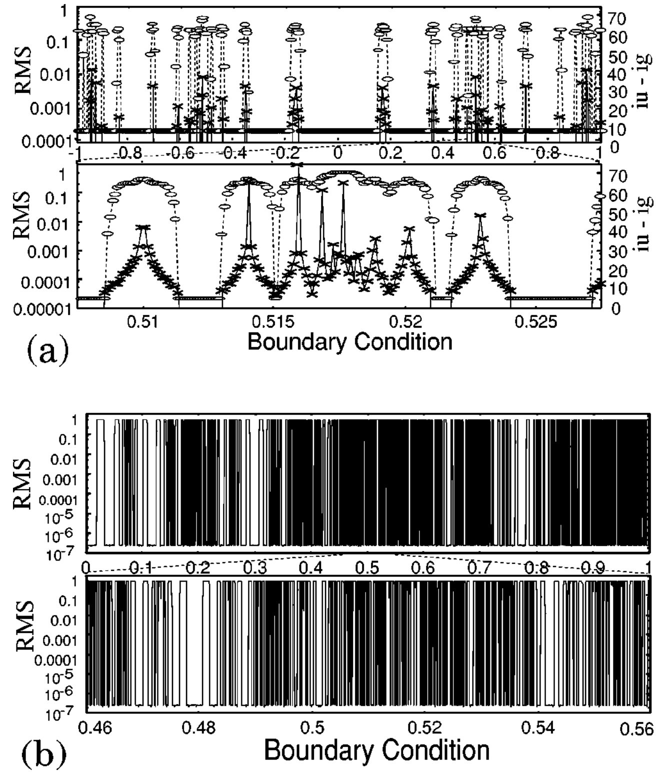

FIG. 4. (a) shows BCD of the RMS of x^{200} (○) and $i_u - i_g$ ($\times$), with $\epsilon = 0.9108$, spatial period = 4, and $\sigma = 2 \times 10^{-4}$. (b) shows BCD of the RMS of x^{200} with $\epsilon = 0.928198$, spatial period = 16, and $\sigma = 2 \times 10^{-8}$. For each, the lower figure is the blow-up of the upper figure, and each figure represents 5000 data points taken at equal intervals.

We now discuss the origin of such sensitive BCD. As seen from Fig. 4, this BCD is due to the complicated structure of the BCD of $i_u - i_g$. Here, i_g has a rather smooth dependence on x^0, and the complicated structure is due mainly to i_u. In fact, there are (infinitely) many local

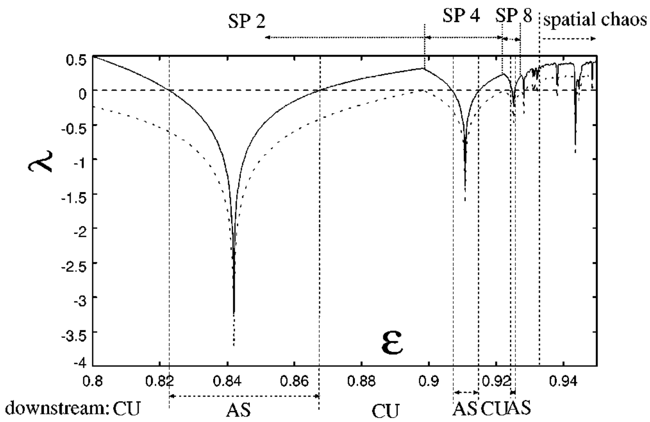

FIG. 5. Lyapunov exponent of the spatial map in Eq. (2) (dotted line) and the spatial Lyapunov exponent averaged over some range of lattice points $\Sigma_{\ell=0}^{15}\lambda^S(i-\ell)/2$ $(i\gg 1)$ (solid lines) are plotted as a function of ϵ for the noiseless case $\sigma=0$. Here, temporally fixed points are spatially period 2 (SP2) for $\epsilon<0.898$, spatially period 4 (SP4) for $0.898<\epsilon<0.922$, spatially period 8 (SP8) for $0.922<\epsilon<0.926$, and spatial chaotic for $\epsilon>0.931$. The spatial Lyapunov exponent shows that the attracted spatial pattern SP2 is absolutely stable (AS) for $0.822<\epsilon<0.867$, and the SP4 pattern is AS for $0.907<\epsilon<0.913$, while the spatial chaos at $\epsilon>0.931$ is always convectively unstable. There are separated parameter regions in which the downstream is absolutely stable.

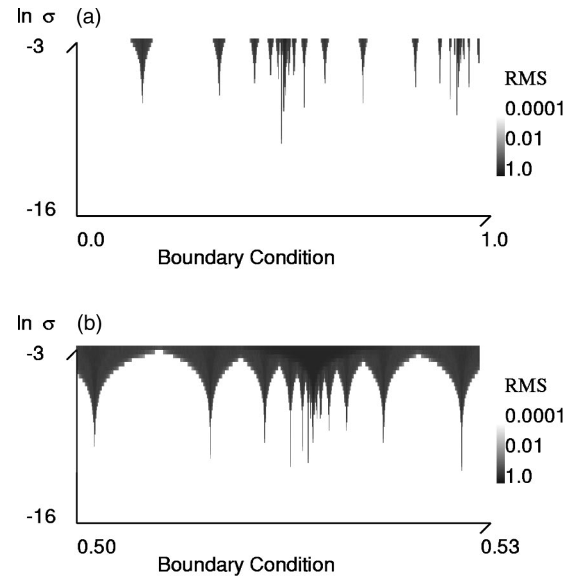

FIG. 6. The RMS of downstream dynamics is plotted as a function of noise strength σ (vertical axis) and boundary condition x^0 (horizontal axis), with $\epsilon=0.9108$. The value of the RMS is plotted by the gray scale, where the darkest pixel shows the case with RMS~ 1, and the brightest the value less than 10^{-4}. (b) is a blow-up of (a). For $\sigma\sim 10^{-16}$, the downstream dynamics is a fixed point for any boundary value of x^0. On the other hand, for a large value of σ around 10^{-3} some tongues merge with each other as shown in (b).

maxima of i_u considered as a function of x^0, in analogy to the plot of i_u-i_g in Fig. 4(a).

Recall that the scale i_u is highly correlated with the duration of the transient process of the spatial map Eq. (2), i.e., the number of steps required for an orbit. It is generated by the dynamics of the map starting from x^0 and falling into a periodic attractor. When the period of the attractor of the map $g(x)$ is L, there are L stable fixed points for the map $x\rightarrow g^L(x)$. Each stable fixed point corresponds to a different phase of the periodic attractor of the spatial map $g(x)$. For each value of the initial condition x^0 in the spatial map, the fixed point to which the $g^L(x)$ map is attracted [i.e., the phase of the cycle in the map $g(x)$] is different.

When multiple attractors coexist, it is often the case that the basin structure for each attractor is fractal [15]. This is true for $g^L(x)$, and here there can be infinitely many basin boundary points. For the spatial period 2 case, there are 4 such points, while there are infinitely many points and the basin boundary is fractal for the case of spatial period 4 (or higher). Note that at unstable fixed points of $g^L(x)$, the phase of the attracted cycle slips. Successive preimages of such fixed points are nothing but the basin boundary of the map $g^L(x)$, with which fractal basin boundary is formed.

For each basin boundary point, the number of transient steps before the attraction of an orbit to an attractor diverges. Hence, i_u takes a local maximum at each point. These points correspond to the local minima of the tongues in Fig. 6. Accordingly, there are infinitely many local maxima of i_u, organized in a self-similar manner. Now, the sensitive dependence on x^0 is understood resulting from a fractal basin boundary in the spatial map. Since this formation of a fractal basin boundary is rather common in one-dimensional maps (with topological chaos), sensitive BCD is expected to be a general phenomenon. It should be stressed that although the mechanism here is based on the fractal basin in the spatial map, the sensitive dependence is explained as a dependence on the *boundary condition* rather than the initial condition [16].

IV. CONCLUSION

In conclusion, we have demonstrated the existence of a sensitive BCD of noise-sustained structure in a system char-

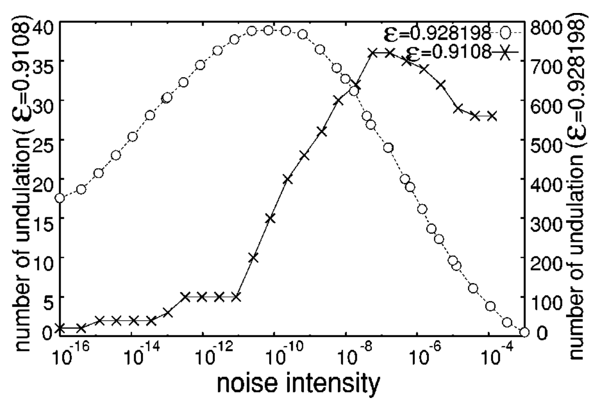

FIG. 7. The noise strength dependence of the number of the undulation in BCD. We have computed the BCD given in Fig. 4 for each noise intensity, using 10000 data points taken at equal intervals, and then counted the number of undulation (ups and downs) in the figure. Plotted are the data for the spatial period 4 (solid line with $\times$) and 16 (dotted line with $\bigcirc$), corresponding to Fig. 4.

acterized by convective instability in the upstream. The origin of boundary condition sensitivity is found to be the complex transient dynamics in the spatial map, which lead to a fractal basin boundary. Accordingly, the sensitive BCD is generally expected as long as the spatial map exhibits topological (or transient) chaos. The analysis presented here can be extended to the case in which, in the absence of noise, the downstream does not possess fixed points, but, rather possesses a stable cycle.

Although we have studied the sensitive BCD for the simplest case with a CML, our analysis can be straightforwardly extended to systems of coupled ODEs and of partial differential equations. It can also be extended to a system with a bi-directional coupling, as long as there is spatial asymmetry. Convective instability in open flow is observed in a wide variety of systems with fluctuations, including chemical reaction networks [12,17], optical networks [18], traffic flow, neural network [19], and open fluid flow. Sensitive BCD is expected to be observed in an such systems. In particular, sensitive BCD in chemical reaction networks may be important in understanding diverse responses in signal transduction systems of cells.

ACKNOWLEDGMENT

This work is partially supported by Grants-in-Aid for Scientific Research from the Ministry of Education, Science, and Culture of Japan (11CE2006 and 11837004).

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