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Globally Coupled Chaos Violates the Law of Large Numbers but Not the Central-Limit Theorem

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The title statement is numerically shown for a globally coupled chaotic system. With an increasing number of elements N, the distribution of the mean field approaches a Gaussian distribution, but the decrease of its mean-square deviation with N stops for large N. This violation of the law of large numbers is found to be caused by the emergence of a subtle coherence among elements, as is measured by the mutual information. With the inclusion of noise, the law of large numbers is restored. The mean-square deviation decreases in proportion to $N^{-\beta}$ with an exponent $\beta < 1$ depending on the noise strength.

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Dynamical systems with global coupling have to be explored in detail as a novel, important field. This class of dynamical systems has been seen in broad branches of science.[1,2] Coupled nonlinear oscillators (pendula) with global feedback give a good model for a Josephson-junction array or charge-density wave with constant electric current. Vortex dynamics in fluids has a long-ranged nonlinear interaction. Globally coupled dynamical systems are also seen in evolutionary dynamics and economics.

Such dynamical systems also provide a novel complex system for biological information processing. In neurodynamics, even a single neuron and a small ensemble of neurons[3] are known to exhibit chaotic behavior. Most of the neural-network studies, however, have adopted very simple dynamical elements. From the standpoint of statistical physics, it is natural and important to study a model with complex dynamical elements (with a chaotic response) and global couplings as an abstraction from neurodynamics.

In this Letter, we study a globally coupled map (GCM). It is a dynamical system of N elements with discrete time. The dynamics consists of local mappings and interactions among all the elements. The GCM has originally been introduced[1,2] as a mean-field-type extension of coupled map lattices (CML).[4,5]

Here we focus on the following form of the GCM:[1]

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{N}\sum_{j=1}^{N}f(x_n(j)), \qquad (1)$$

where n is a discrete time step and i is the index of elements $(i=1,2,\ldots,N)$. The function $f(x)$ is chosen to be the logistic map

$$f(x)=1-ax^2, \qquad (2)$$

since it has been thoroughly investigated as a prototype of dissipative chaos.

Our GCM (1) has a remarkably rich behavior, partly similar but much richer than the mean-field model for the spin glass by Sherrington and Kirkpatrick.[6] It exhibits successive phase transitions among coherent, ordered, intermittent, and turbulent phases, as the nonlinearity is increased. Attractors with dynamical tree structures are found in the intermittent phase. Coding and bifurcation of the attractors are discussed in Ref. 1, as well as switching among them.

Our model has two conflicting tendencies; destruction of coherence by chaotic instability in each element, and synchronization by global averaging. If the nonlinearity a is large enough, the former tendency wins and none of the elements are synchronized.

In the present Letter we study this turbulent state, where any coherence is completely destroyed by chaos. All elements take different values, without any explicit symptom of correlation among elements. Neither intermittent time series nor any δ peaks in the power spectrum are observed. All of the Lyapunov exponents are positive.[7]

Let us consider the fluctuation of the mean field $(1/N)\sum_i f(x(i))$. Since the $x(i)$ take random values al-

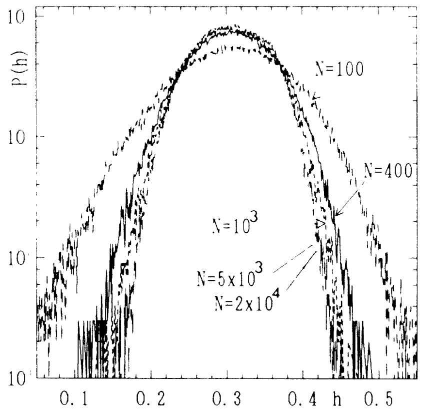

FIG. 1. Distribution function of mean field h, for $N=100$, 400, 10^3, 5×10^3, and 2×10^4. Histograms for the last two sizes agree within statistical error. $a=1.99$ and $\epsilon=0.1$. The histogram is obtained from a sampling over 10^5 time steps, after discarding the initial 10^4 steps, using a bin width 10^{-3}.

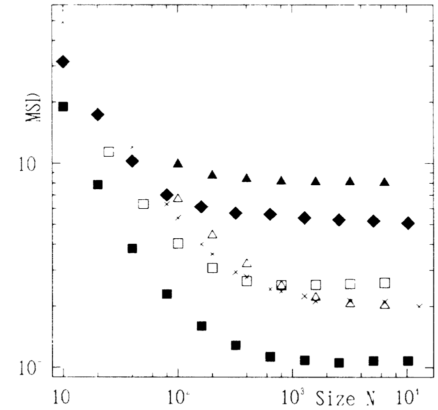

FIG. 2. Mean-square deviation (MSD) of the distribution of mean field h, plotted as a function of system size. The MSD is calculated over 10^5 time steps after 10^4 transients. $\epsilon=0.1$. The parameter a is 1.80 (■), 1.83 (□), 1.85 (◆), 1.92 (▲), 1.95 (△), and 1.99 (×).

most independently in the turbulent phase, one might expect that the aggregate

$$h_n \equiv (1/N)\sum_j f(x_n(j)) \tag{3}$$

obeys the law of large numbers and the central-limit theorem. If this were true, the mean-square deviation (MSD) of the mean field h would decrease with N as N^{-1}. The interaction term in (1) could be replaced by a noise whose root mean square is $O(1/\sqrt{N})$. Then, in the limit of $N\to\infty$, the dynamics (1) would reduce to N-independent logistic maps given by $x_{n+1}(i)=(1-\epsilon)\times f(x_n(i))+\epsilon C$, with a constant mean field $C=(1/N)\times\sum_i f(x(i))$.

To examine the above expectation, we have numerically measured the distribution of the mean field h_n. As shown in Fig. 1, the distribution function $P(h)$ agrees with a Gaussian form, when N is large. Thus the central-limit theorem is valid here.

For the verification of the law of large numbers, we plot the MSD of the mean field $(\delta h)^2 \equiv \langle h^2\rangle - \langle h\rangle^2$, where $\langle \cdots \rangle$ is the average over the distribution $P(y)$; $\langle A\rangle \equiv \int P(h)A(h)dh$. If each element $x(i)$ is approximated by an uncorrelated random number, it is expected that $(\delta h)^2 \propto 1/N$.

In Fig. 2, the MSD is plotted with the change of N. The decrease of the MSD with N stops around a crossover size $N=N_c(a)$.[8] (N_c depends on a.) This observation means that *globally coupled chaos violates the law of large numbers, but not the central-limit theorem.*

The existence of a size-independent fluctuation suggests the emergence of some order in our dynamics. First, the dynamics of the mean field h_n is studied. Although its motion is aperiodic, there are broad peaks in the power spectrum for h_n. The peaks get sharper with increasing of N up to N_c (see Fig. 3), while the spectrum is invariant under a change of N for $N>N_c$. This sharpening of peaks suggests the emergence of a partly coherent motion. In the power spectrum of $x_n(i)$ for a single element i, such sharpening of peaks is not observed with an increase of the size.

Second, we compute the mutual information among elements to measure the correlation. The application of mutual information to dynamical systems has been pioneered by Shaw,[9] and has been extended to spatiotemporal chaos.[10] For the calculation of the mutual information, we introduce a single-point probability $P_i(y)$ that $x(i)$ takes the value y, and a two-point joint probability $p_{i,j}(y,z)$ that $x(i)$ takes the value y and $x(j)$ takes the value z $(i\neq j)$. The mutual information $\mu_{i,j}$ is defined by

$$\mu_{i,j} = -\int\int \ln\frac{p_{i,j}(x,y)}{P_i(x)P_j(y)}\,dx\,dy\,.$$

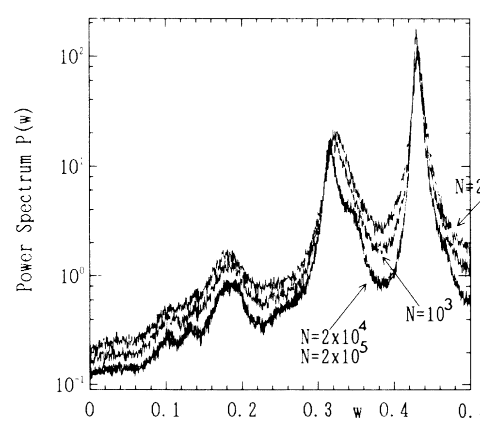

FIG. 3. Power spectra of the time series of mean field h_n, for $N=200$, 10^3, 2×10^4, and 2×10^5. The last two spectra agree within the accuracy. They are calculated from 100 sets of 1024-step time series (in total, 1024×100 steps after 10^4 transients).

Here the probability is defined through a long-time sampling. In the turbulent phase, however, ergodicity is found to hold numerically. Thus the above probability functions and the mutual information are independent of elements i. To increase the sampling size, we can introduce both temporal sampling and averaging over the elements. Hence the mutual information is computed by

$$\mu = -\frac{2}{N(N-1)} \int\int \sum_{i>j} \ln p_{i,j}(x,y)dx\,dy + \frac{2}{N}\int \sum_i \ln P_i(x)dx\,. \tag{4}$$

In Fig. 4, we have plotted μ with the change of size N. As is seen, there remains a finite mutual correlation even when N gets large. Although the correlation is very small (the order of 10^{-3}), it is distinguished from 0. Indeed, μ is less than the order of 10^{-5} within the sampling of the same time interval, if a small noise is applied to our system. The remaining finite correlation among elements can cause the breakdown of the law of large numbers.

In the intermittent phase, similarity with the spin-glass phase has been pointed out in Ref. 6. To search for a possible frozen order as in the spin glass, we introduce a quantity similar to the Edwards-Anderson order parameter for the spin glass.[6] For this, we define the following relative closeness $S_n^{i,j}$ between two elements:

$$S_n^{i,j} = \begin{cases} 1 & \text{if } |x(i)-x(j)| < \delta, \\ 0 & \text{otherwise,} \end{cases}$$

where δ is a precision to judge the relative closeness. We study the following temporal correlation function:

$$C(t) = \sum_{i,j} (\langle S_{m+t}^{i,j} S_m^{i,j}\rangle - \langle S_m^{i,j}\rangle^2)\,. \tag{5}$$

In our simulation for the turbulent state, $C(t)$ decays to zero exponentially with time, if the precision δ is not large (<0.5). Thus there is no relationship of freezing between two elements.

Is the law of large numbers recovered with the addition of noise? To answer this question, we simulate the model

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j)) + \sigma\eta_n^i\,,$$

with an uncorrelated random number η_n^i, homogeneously distributed over $[-1,1]$. In Fig. 5, the MSD ($=\langle\delta h^2\rangle$) is plotted as a function of size N. If the noise strength σ is larger than a threshold σ_c (≈ 0.004 for the parameters in the figure), the MSD decreases with the size N, implying the recovery of the law of large numbers. What is striking here, however, is that the MSD decreases with size according to $\langle\delta h^2\rangle \propto N^{-\beta}$ with $\beta < 1$. The exponent β approaches unity with increasing σ.

This kind of anomalous power-law dependence may originate in the hierarchical structure of our attractor discussed in Ref. 1, since the diffusion in a treelike structure shows an anomalous power-law dependence.[11]

To conclude, the law of large numbers is violated in our globally coupled chaotic system. Its origin is ascribed to the emergence of a subtle correlation among elements. To confirm the universality of our results, we have also studied GCMs with a local tent map [$f(x) = a(0.5-|x-0.5|)$] and a circle map [2,1]. Again, the fluctuation of the mean field does not decrease with the size, implying a violation of the law of large numbers.

Our conclusion is in contrast with fluctuations in a short-ranged CML, where a finite correlation length ξ

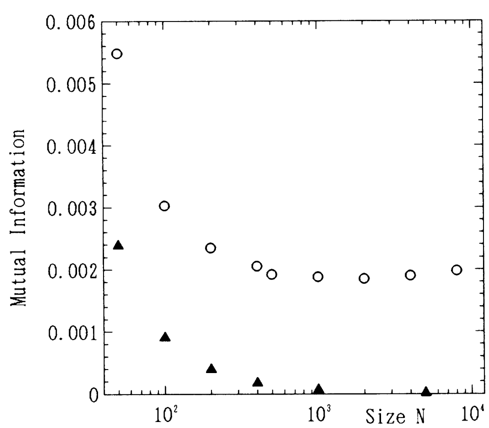

FIG. 4. Two-point mutual information. Calculated from $2\times10^7/N$ time steps after discarding 10^4 transients. Obtained from GCMs without noise (○), and with random numbers homogeneously distributed over $[-0.01,0.01]$ (▲). $a=1.99$ and $\epsilon=0.1$.

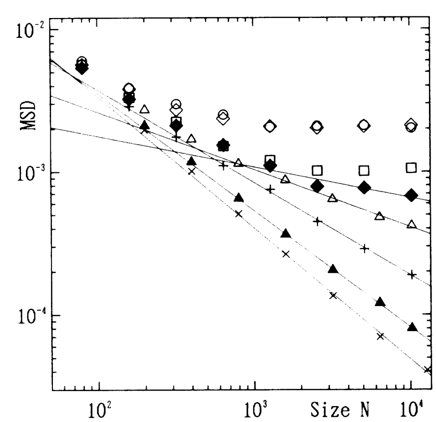

FIG. 5. Mean-square deviation of the distribution of mean field h, in our GCM with the addition of noise, plotted as a function of system size. $a=1.99$ and $\epsilon=0.1$. The noise strength σ is 0.02, 0.01, 0.007, 0.005, 0.0045, 0.004, 0.0008, and 0.0001 from bottom to top. For $\sigma>\sigma_c\approx0.004$, MSD decays as $N^{-\beta}$. The power β is estimated to be 0.92, 0.82, 0.65, 0.40, and 0.02 for $\sigma=0.02$, 0.01, 0.007, 0.005, and 0.0045, respectively. The MSD is calculated over 10^5 time steps after 10^4 transients.

exists for almost all parameters. The existence of ξ leads to a decrease of the averaged mutual information μ [Eq. (4)], in proportion to ξ/N. In a globally coupled dynamical system, a correlation *length* is not defined, and a finite ratio of correlation may remain even for large N. If this scenario is valid, the conclusion of the present Letter may be a general feature in globally coupled chaos.

A detailed study of the mechanism and universality of our results will be reported elsewhere, as well as the construction of the thermodynamics for globally coupled chaos.[12]

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