Collective Chaos

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An algorithm to characterize collective motion as the orbital instability at a macroscopic level is presented, including the introduction of "collective Lyapunov exponent." By applying the algorithm to a globally coupled map, existence of low-dimensional collective chaos is confirmed, where the scale of (high-dimensional) microscopic chaos is separated from the macroscopic motion, and the scale approaches zero in the thermodynamic limit. [S0031-9007(98)07685-6]

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Low-dimensional chaotic motion often arises from a system with many degrees of freedom. A classical example is chaos in a fluid system (such as Rayleigh-Bénard convection), where very high-dimensional chaotic motion should underlie at a molecular scale. A canonical answer for the condition to have low-dimensional chaos at a macroscopic level is given by separation of scales distinguishable from a microscopic level. Still it is not clear how such separation is possible, since chaos can lead to the amplification of a small-scale error.

To address the question, we consider a certain dynamical system that shows some lower-dimensional motion for a certain macroscopic variable (e.g., average of microscopic variables), whereas (microscopic) variables keep high-dimensional chaos. There the number of positive Lyapunov exponents is proportional to the system size, and diverges in the "thermodynamics limit" (infinite system size limit). In this Letter, in order to characterize such macroscopic motion, Lyapunov exponent at a macroscopic scale is introduced, which specifies the growth rate of error at macroscopic variables. By studying the dependence of the exponent on the length scale in phase space and the system size, it is shown how the "collective chaos" is compatible with microscopic chaos, and how they are separated at the thermodynamic limit. Here chaos in the variables of the dynamical system is referred to as "microscopic" chaos.

First note that the conventional Lyapunov exponents for the dynamical system are not relevant to the characterization of collective motion. In order to calculate the Lyapunov exponent for the collective motion, an infinitesimal limit of disturbance to a trajectory should be taken at a "macroscopic" level. Rigorously speaking, the macroscopic level appears in the thermodynamic limit (system size $N \to \infty$). Thus, it is necessary to take the thermodynamic limit first and then the limit of disturbance scale, to characterize the collective dynamics. However, in the conventional computation of the Lyapunov spectrum we first take the infinitesimal limit of disturbance applied to the orbit, and see the asymptotic behavior of the spectrum in the thermodynamic limit. Hence, the exponent cannot characterize the collective motion. This problem can be resolved by noting the order of limit to define the Lyapunov exponent.

Since we are concerned with a system of a large but finite size, the above order of limit implies that we have to keep the disturbance amplitude finite, so that the disturbance is studied at a macroscopic level. (Roughly speaking the disturbance at a macroscopic variable should be larger than $1/\sqrt{N}$.) To study such orbital instability, the finite-size Lyapunov exponent introduced by Vulpiani *et al.* [1] is useful. It is given by

$$\lambda_{\delta_0}(\Delta) = \left\langle \frac{1}{\tau} \log \frac{\Delta}{\delta_0} \right\rangle, \tag{1}$$

where τ is the maximum time such that $|x'_n - x_n| < \Delta$ for trajectories x_n and x'_n starting from x_0 and $x'_0 = x_0 + \delta_0$ respectively, while $\langle \cdot \rangle$ is an average over the trajectories starting from different initial values. The length scale Δ can be considered as the scale of observation.

Here we consider measurement of the finite-size Lyapunov exponent for macroscopic variables with a certain finite-size disturbance at a macroscopic level. As long as the system size is finite, this finite-size Lyapunov exponent reflects not only the macroscopic motion but also the microscopic chaos. On the other hand, if low-dimensional macroscopic dynamics has a characteristic time scale separated from the microscopic dynamics, it will be possible to extract the growth rate of perturbation in the collective motion from the finite-size Lyapunov exponent for the macroscopic variable(s). To do so, we postulate the following assumptions that are expected to hold if the collective dynamics is low-dimensional chaos or on a torus.

First note that in the limit $\Delta \to 0$ and $\delta_0 \to 0$, the finite-size Lyapunov exponent $\lambda_{\delta_0}(\Delta)$ for macroscopic variable in finite system size is expected to converge to the maximum Lyapunov exponent λ_m, which is determined by the conventional Lyapunov exponents for the microscopic variables directly.

Considering that the collective dynamics appears by coarse-grained macroscopic variables, we postulate that there are length scales (in the phase space) $\Delta \in [\Delta_m, \Delta_C]$, where the macroscopic variable is characterized by "collective Lyapunov exponent" λ_C. Below

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$\Delta < \Delta_m$ the microscopic chaos dominates, while the orbit is out of the attractor for $\Delta > \Delta_C$ at a macroscopic level. To have low-dimensional collective dynamics, it is postulated that λ_C is independent of N (as long as it is large enough), and that Δ_m should approach zero with $N \to \infty$ while Δ_C remains finite.

Based on the above assumptions, we can have a form of the finite-size Lyapunov exponent as a function of the scale Δ. Let δ_n denote the distance from the original trajectory at time step n. For the scale $\Delta < \Delta_m$, δ_n increases proportionally with $\exp(\lambda_m n)$. Hence $\tau(\Delta) = \frac{1}{\lambda_m} \log \frac{\Delta}{\delta_0}$ follows, independently of the collective dynamics.

On the other hand, for the scale $\Delta_m < \Delta < \Delta_C$, δ_n is given as $\delta_n \propto \exp(\lambda_C n)$ for a chaotic case with $\lambda_C > 0$, or $\delta_n \propto n^\kappa$ for a torus case with a certain constant κ. Corresponding to each collective motion, $\tau(\Delta)$ and the finite-size Lyapunov exponent $\lambda_{\delta_0}(\Delta)$ are given by

$$\tau(\Delta) = \begin{cases} \frac{1}{\lambda_C} \log \frac{\Delta}{\Delta_m} + \frac{1}{\lambda_m} \log \frac{\Delta_m}{\delta_0} & \text{(chaos)}, \\ C(\frac{\Delta}{\Delta_m})^{1/\kappa} + \frac{1}{\lambda_m} \log \frac{\Delta_m}{\delta_0} & \text{(torus)}, \end{cases} \tag{2}$$

and

$$\lambda_{\delta_0}(\Delta) = \begin{cases} \dfrac{\lambda_m \lambda_C \log \frac{\Delta}{\delta_0}}{\lambda_C \log \frac{\Delta_m}{\delta_0} + \lambda_m \log \frac{\Delta}{\Delta_m}} & \text{(chaos)}, \\ \dfrac{\log \frac{\Delta}{\delta_0}}{\frac{1}{\lambda_m} \log \frac{\Delta_m}{\delta_0} + C(\frac{\Delta}{\Delta_m})^{1/\kappa}} & \text{(torus)}, \end{cases} \tag{3}$$

where Δ_m, and λ_C, or κ and C are fitted parameter to data, λ_m is the maximum Lyapunov exponent, and δ_0 is the value of initial disturbance. In order to obtain the values of parameters Δ_m, and λ_C, or κ and C easily, it is convenient to transform Eq. (3) to remove δ_0 dependence of the data. For it, we define $t(\Delta)$ as $t(\Delta) = \tau(\Delta) + \frac{1}{\lambda_m} \log \delta_0$, which characterizes the time for amplification of error from a certain scale independent of δ_0. From Eq. (2), we obtain

$$t(\Delta) = \begin{cases} \frac{1}{\lambda_C} \log \Delta + (\frac{1}{\lambda_m} - \frac{1}{\lambda_C}) \log \Delta_m & \text{(chaos)}, \\ C(\frac{\Delta}{\Delta_m})^{1/\kappa} + \frac{1}{\lambda_m} \log \Delta_m & \text{(torus)}. \end{cases} \tag{4}$$

From data, we can easily obtain t-Δ plot (t-$\log \Delta$ or $\log t$-$\log \Delta$ plot), in which Δ_m appears as a shift of constant, and λ_C or κ is given by a slope in a suitable plot. In order to confirm the existence of the low-dimensional collective motion, it is necessary that Δ_m decreases with N as $1/\sqrt{N}$ for a constant λ_C.

To demonstrate our method and to show the existence of some lower-dimensional macroscopic motion, we study a certain coupled dynamical system, which shows collective motion [2–7]. Here we adopt a "heterogeneous" globally coupled map (GCM) with a distributed parameter:

$$x_{n+1}(i) = (1 - \epsilon) f_i(x_n(i)) + \frac{\epsilon}{N} \sum_{j=1}^{N} f_j(x_n(j)), \tag{5}$$

where $x_n(i)$ is the variable of the ith element ($i = 1, 2, 3, \ldots, N$) at discrete time n, and $f_i(x)$ is an internal dynamics for each element. For the internal dynamics we choose the logistic map $f_i(x) = 1 - a(i)x^2$, where the parameter $a(i)$ for the nonlinearity is distributed between $[a_0 - a_w, a_0 + a_w]$ as $a(i) = a_0 + \frac{a_w(2i-N)}{N}$. (In the following, the parameters are indicated by $a = a_0 \pm a_w$.) As a macroscopic variable, we adopt the mean field,

$$h_n = \frac{1}{N} \sum_{i=1}^{N} f_i[x_n(i)], \tag{6}$$

in which the collective motion is contained.

The study of collective motion in GCM has gathered much attention [2–5]. When the coupling ϵ is small enough, oscillation of each element is mutually desynchronized, and the effective degrees of freedom increase in proportion to the number of elements N. Still, a macroscopic variable is found to show some kind of ordered motion distinguishable from noise, ranging from torus to high-dimensional chaos [3–5].

For instance, Fig. 1 gives a return map of the mean field dynamics of the GCM (5), which shows some pattern that may suggest low-dimensional chaos. Torus motion is also found for some parameter values [5]. In these cases, microscopic motion keeps high-dimensional chaos, i.e., all of the N Lyapunov exponents are positive, even if there appears quasiperiodic motion for the collective variable h_n as N goes to infinity [5]. Here we demonstrate the existence of low-dimensional collective motion by the above collective Lyapunov exponent λ_C and by the N dependence of Δ_m.

Figure 2 gives the finite-size Lyapunov exponent for the mean field dynamics of the GCM (5). Here we perturb the orbit to give rise to a change from h_0 to $h_0' = h_0 + \delta_0$ (see the caption of Fig. 2 for a detailed description). In Fig. 3, t is plotted as a function of Δ. As is shown in Fig. 3(a), the slope of the semilog plot is independent of N. The Lyapunov exponent λ_C, characterizing the collective motion, is given by the inverse of the slope,

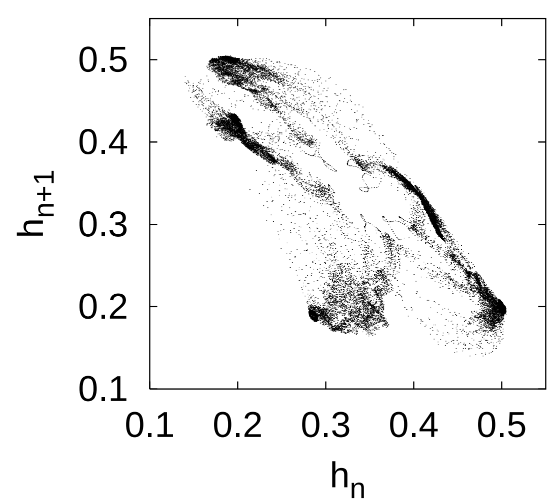

FIG. 1. An example of return map for chaotic collective motion. $a = 1.92 \pm 0.044, \epsilon = 0.1, N = 10^7$. Points (h_n, h_{n+1}) are plotted over 3×10^4 steps after transient are discarded.

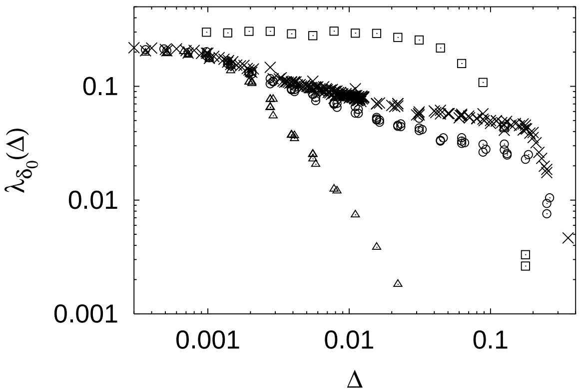

FIG. 2. $\lambda_{\delta_0}(\Delta)$ is plotted for the model (5), (⊙) $a_0 = 1.92 \pm 0.044$, $\epsilon = 0.1$; (×) $a = 1.9 \pm 0.025, \epsilon = 0.098$; (△) $a = 1.9 \pm 0.025, \epsilon = 0.11$; (⊡) $a = 1.69755 \pm 0, \epsilon = 0.008$). $N = 10^7$. Initial perturbation amplitude δ_0 is fixed at 1.0×10^{-7}. For computation, displacement $h_0' = h_0 + \delta_0$ is created by perturbing the orbit as $x_0'(i) = x_0(i) + \frac{1}{N}\delta_0 \times \sigma$, where σ is a random number in $[-1, 1]$. Each point is obtained by averaging over 100 samples. Specific choice of this perturbation scheme is irrelevant to our results, as long as the collective variable is perturbed. Adopting the algorithm to be presented, the collective motion is shown to be torus (△), low-dimensional chaos (⊙ and ×), and high-dimensional chaos (⊡).

and is estimated as 0.02, which is much smaller than the maximum Lyapunov exponent of the system (see the caption of Fig. 4). Note also that no plateau is visible in the finite-size Lyapunov exponent in Fig. 2 corresponding to λ_C. On the other hand, Δ_m, given by the shift of the plots, decreases with N, while Δ_C does not show significant change [8]. Thus the scale for the collective motion $\Delta_m < \Delta < \Delta_C$ increases with N. In Fig. 4, N dependence of Δ_m is plotted, which gives $\Delta_m \sim \frac{1}{\sqrt{N}}$, whose form is expected from the central limit theorem. Hence the emergence of low-dimensional collective chaos at the thermodynamic limit is confirmed.

We have also applied the present algorithm to the case with a collective torus motion. Figure 3(b), (t-Δ plot), shows that κ, the inverse of the slope, is 0.5, independent of N. Indeed, this exponent $1/2$ is expected from the diffusion of phase on the torus. The decrease of Δ_m with N is also plotted in Fig. 4, which again shows the expected decrease of $\Delta_m \sim \frac{1}{\sqrt{N}}$. Hence the collective torus motion is demonstrated.

In this Letter, we have proposed an algorithm to characterize the collective (chaotic) motion, and applied to it to a GCM. We have introduced the collective Lyapunov exponent, to characterize the growth rate of perturbation in the collective motion. The microscopic chaotic motion exists at a small scale of the macroscopic variable, but such scale Δ_m is shown to decrease as $1/\sqrt{N}$. Hence, the macroscopic motion is separated from the microscopic motion and the emergence of low-dimensional collective motion with $N \to \infty$ is confirmed [9].

Existence of low-dimensional collective chaos in the presence of microscopic chaos has often been suspected

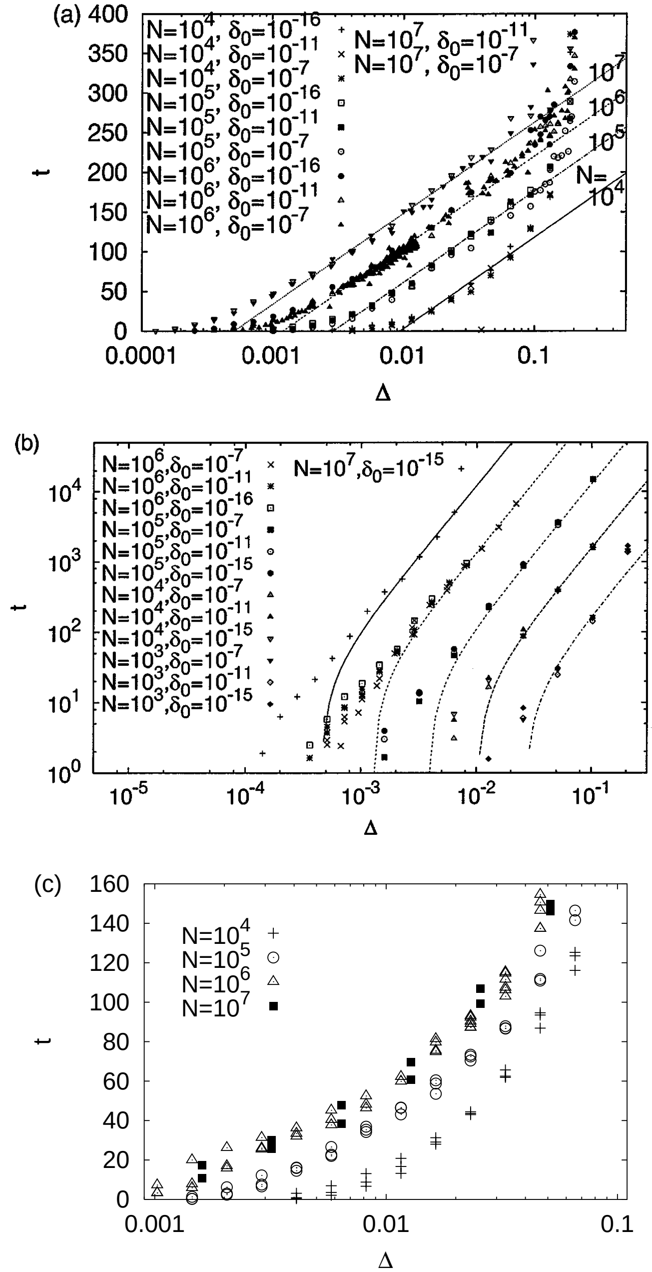

FIG. 3. The normalized time steps $t(\Delta)$ are plotted for $N = 10^4$, 10^5, 10^6, and 10^7, with the fitted curves Eq. (4). (a) *Chaotic case* (with a semilog plot): for $a = 1.9 \pm 0.025, \epsilon = 0.098$. (b) *Torus case* (with a log-log plot): for $a = 1.9 \pm 0.025, \epsilon = 0.11$. The maximum Lyapunov exponent $\lambda_m = 0.41$ (a), 0.39 (b) are obtained directly from the GCM (5). The parameters obtained by a least-square fitting algorithm give (a) $\lambda_C = 0.02$, and (b) $\kappa = 0.5$. (c) *High-dimensional case*, which does not obey Eq. (4), (with a semilog plot): for $a = 1.6962 \pm 0, \epsilon = 0.008$. In this case, while the return map shows some structure, t for $N = 10^6$ and 10^7 are not separated any more. For (c), the data from $\delta_0 = 10^{-7}, 10^{-11}, 10^{-16}$ are plotted by the same symbol, since the difference by δ_0 is not observed as in (a) and (b).

[10]. Indeed, for a GCM with homogeneous elements (i.e., with $a = a_0 \pm 0$), such low-dimensional collective chaos has not been observed, and the collective motion there is

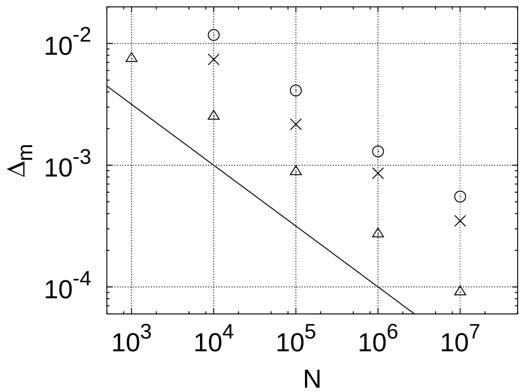

FIG. 4. The microscopic length scales Δ_m are plotted as a function of N for several parameters. Δ_m is obtained from the fitting indicated in Fig. 3. The parameters are as follows. (⊙) $a = 1.92 \pm 0.044, \epsilon = 0.1$ (chaos: $\lambda_C = 0.009$, $\lambda_m = 0.42$); (×) $a = 1.9 \pm 0.025, \epsilon = 0.098$ (chaos: $\lambda_C = 0.02$, $\lambda_m = 0.41$); and (△) $a = 1.9 \pm 0.025, \epsilon = 0.11$ (torus: $\kappa = 0.5$, $\lambda_m = 0.392$).

believed to be high dimensional [4,5]. In Fig. 3(c), we have also applied our algorithm to this case. The separation of scales is not clear, and the data cannot be fitted with (6). The shift of the plot gets smaller with the increase of N. At least Δ_m does not decrease as $1/\sqrt{N}$ [11].

The t-Δ plot provides a tool to distinguish low-dimensional collective chaos from high-dimensional collective chaos. In the former case, the plot shifts as $\log(\sqrt{1/N})$ with N, while for the latter case such a shift is not observed. This distinction generally holds, even if the approximation to get Eq. (4) may not be very good [12].

Our present algorithm to extract macroscopic motion is applicable to any system subjected to microscopic chaos, including a coupled oscillators system [7], spatially extended systems from coupled map lattice to partial differential equations. It is also expected to be applied even if we do not know the equation of motion, since the method of [1] is based on Wolf's algorithm [13] developed for the estimate of Lyapunov exponents from experimental data. Thus, we hope that our method developed in this Letter is applicable to data obtained from experiments.

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Note added.—After the completion of the present manuscript, the authors are informed of the recent preprint by M. Cencini, M. Falcioni, D. Vergni, and A. Vulpiani [12], where a related study to the collective chaos is presented.

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[9] The existence of low-dimensional torus motion is detected, as a plateau of the correlation dimension within a certain range of scale. In the case of collective chaos, the plateau is not clearly visible in the plot, although the plot may suggest some signature of low dimensionality.
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