Noiseless Collective Motion out of Noisy Chaos

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We consider the effect of microscopic external noise on the collective motion of a globally coupled map in fully desynchronized states. Without the external noise a macroscopic variable shows high-dimensional chaos distinguishable from random motions. With the increase of external noise intensity, the collective motion is successively simplified. The number of effective degrees of freedom in the collective motion is found to decrease as $-\log\sigma^2$ with the external noise variance σ^2. It is shown how the microscopic noise can suppress the number of degrees of freedom at a macroscopic level. [S0031-9007(99)09278-9]

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Chaotic motions have been observed experimentally in physical, chemical, and biological systems. Since the evolution of these systems is subjected to external fluctuations, the observability of deterministic chaos depends on how the external fluctuations influence it [1,2]. Motivated by this point, extensive studies have been carried out about the enhancement of predictability and unpredictability of chaotic motions [2,3].

So far such studies are restricted to low-dimensional dynamical systems. Low-dimensional chaotic motion often arises as a macroscopic motion out of microscopic chaos with many degrees of freedom. Let us consider external fluctuations imposed on the microscopic level rather than the macroscopic level, as is probable in natural systems, such as fluid turbulence, or neural systems with a large number of neurons. Since chaos can amplify a small-scale error, it would be natural to ask a question how such a low-dimensional macroscopic chaos is possible out of high-dimensional chaotic systems subjected by external fluctuations.

To address the question, we note that in certain coupled dynamical systems a macroscopic variable shows seemingly low-dimensional motions, while microscopic variables keep high-dimensional chaos. Such a phenomenon has been extensively studied as a collective motion in a coupled map lattice [4], globally coupled oscillators [5], and a globally coupled map [6–14]. In the present Letter, we focus on the effect of noise on the collective motion of a globally coupled map (GCM).

The present GCM consists of N elements iterated by a local dynamics $f(x)$ with a global coupling among elements and an external noise. The dynamics is given by

$$x_{n+1}(i) = (1 - \epsilon) f(x_n(i)) + \frac{\epsilon}{N} \sum_{j=1}^{N} f(x_n(j)) + \xi_n(i), \tag{1}$$

for the ith element at time step n. Here, we adopt the logistic map $f(x) = 1 - ax^2$ for the local dynamics and Gaussian random process for $\xi_n(i)$, with $\langle \xi_n(i) \rangle = 0$ and $\langle \xi_n(i)\xi_m(j) \rangle = \sigma^2 \delta_{nm}\delta_{ij}$. The variance of Gaussian distribution is denoted by σ^2 [15].

In the noiseless case ($\sigma = 0$), if the coupling strength ϵ is small enough, the motion of each element seems to be independent from the others. Even in such cases, the motion in macroscopic variables counterintuitively does not vanish in the thermodynamic limit ($N \to \infty$). This has been studied as "collective motion" in GCM [6–14], which implies some sort of coherence between elements.

Figure 1 gives an example of the collective motion in GCM (1) without the external noise ($\sigma = 0$). We adopt the mean field,

$$h_n = \frac{1}{N} \sum_{i=1}^{N} f(x_n(i)), \tag{2}$$

as a macroscopic observable. While the microscopic motion shows high-dimensional chaos in the sense that the Lyapunov dimension is proportional to the number of elements N, the macroscopic motion shows a quasi-periodic-like structure [16] as is shown in Fig. 1. In almost all the parameter values, the mean-field motion shows some coherent structure ranging from quasi-periodic-like to a higher-dimensional one distinguishable from random motions [12]. However, even if the macroscopic motion looks like quasiperiodic, scattered points around the torus-like structure depicted in Fig. 1 do not vanish even in the

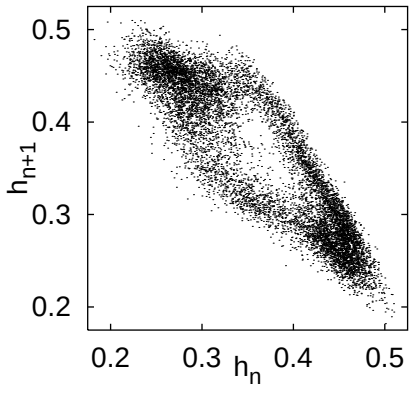

FIG. 1. A return map of the collective motion in GCM (1) without noise. $a = 1.86$, $\epsilon = 0.1$, $\sigma = 0.0$, $N = 10^7$.

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thermodynamic limit [13,14], suggesting high dimensionality of the collective motion. So far, the mean-field motion is considered to be an infinite dimensional motion even when the toruslike structure is observed [10–13].

The addition of noise may, however, destroy such coherence among elements. One of the authors reported that the microscopic external noise leads the mean square deviation (MSD) of the mean-field distribution decreases with N [6]. (The MSD decreases in proportion to $1/N^{\beta}$ with $\beta \leq 1$, when σ is larger than a certain constant.) On the other hand, it is also reported that the external noise sharpens the peak in the power spectrum of the collective motion [8]. In this Letter, we clarify the effect of noise on the collective motion in GCM.

Consider a one-body distribution function $\rho_n(x)$ of the elements to study the behavior of the collective motion in the thermodynamic limit $N \rightarrow \infty$. Since the mean-field value

$$h_n = \int f(x)\rho_n(x)\,dx\,, \tag{3}$$

is applied commonly for each element, and since the additive noise can be represented as a deterministic diffusion process of the distribution function in the thermodynamic limit, the evolution of $\rho_n(x)$ obeys the Perron-Frobenius equation written as

$$\rho_{n+1}(x) = \int dy\, \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-[F_n(y)-x]^2/2\sigma^2} \rho_n(y)\,, \tag{4}$$

with $F_n(x) = (1 - \epsilon)f(x) + \epsilon h_n$.

Figure 2 gives an example of return maps of the mean-field value obtained numerically in GCM with the external noise. The parameters a and ϵ are the same as in Fig. 1. Numerical calculation was carried out through integration of Eq. (4) using a sufficiently large dimensional vector to approximate $\rho_n(x)$. As is shown in Fig. 2, the motion has a clearer structure than the motion without the noise. By increasing σ, lower-dimensional chaos [Fig. 2(a)], locking states, and motions on a torus [Fig. 2(b)], are observed. With the further increase of σ, the collective motion collapses to a fixed point. Hence, with the increase of the noise a sort of bifurcation to lower-dimensional motions is observed.

To clarify this point, it seems natural to measure the number of effective degrees of freedom in the collective motion. We calculate the Lyapunov dimension of the dynamics of $\rho_n(x)$. The Lyapunov exponents are given by the growth rates of tangent vectors around the orbit of Eq. (4). For numerical calculation, $\rho_n(x)$ is approximated by a sufficiently large dimensional vector, and its linear stability around the orbit is studied.

In Fig. 3, the Lyapunov dimension denoted by D_C is plotted as a function of the noise variance σ^2. For sufficiently large σ, only the stationary state is observed and D_C is zero accordingly. With the decrease of σ we have found the low-dimensional collective motion [$D_C \sim O(1)$] ranging from the motion on a torus to low-dimensional chaos. With the further decrease of σ, the dimension grows as

$$D_C \propto -\log\sigma^2. \tag{5}$$

This implies that the number of effective degrees of freedom goes to infinity in the zero noise limit, as is expected from the analysis of the collective motion in GCM without the external noise.

In the large σ regime a variety of bifurcations appears, which may strongly depend on the parameters. However, the above relation (5) generally holds in the high-dimensional collective motions in the small σ regime, as we have examined for several parameters.

Although the evolution rule is originally given for the microscopic variables, our main interest is on the behavior of macroscopic variables which would be the only possible observable in typical cases. Thus, it is highly desirable to obtain a closed description of the behavior of the macroscopic variables, which could be written as

$$h_n = h(h_{n-1}, h_{n-2}, \ldots)\,, \tag{6}$$

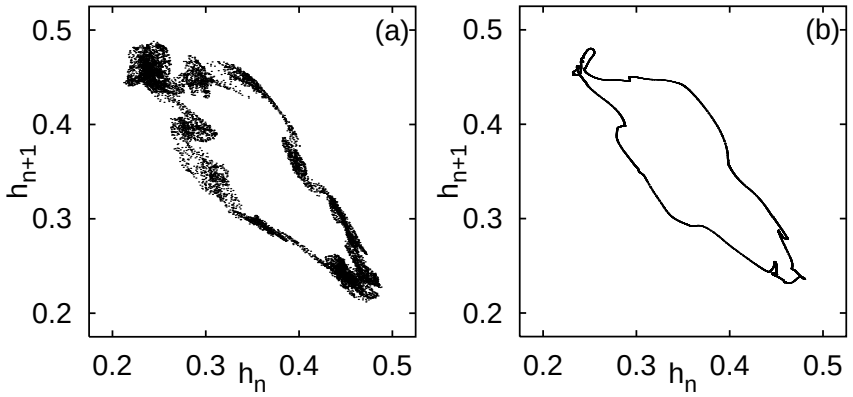

FIG. 2. Return maps of the collective motion in GCM (1) with noise. $a = 1.86$ and $\epsilon = 0.1$. The noise variances are (a) $\sigma^2 = 1.5 \times 10^{-6}$ and (b) $\sigma^2 = 2.7 \times 10^{-6}$. The Lyapunov dimension for (a) is estimated at 3.08. Numerical calculation was carried out with integration of Eq. (4) using a sufficiently large dimensional vector to approximate the distribution function. When the system size in Fig. 1 is sufficiently large and the noise in this figure is small enough, the difference between these are expected not to be significant.

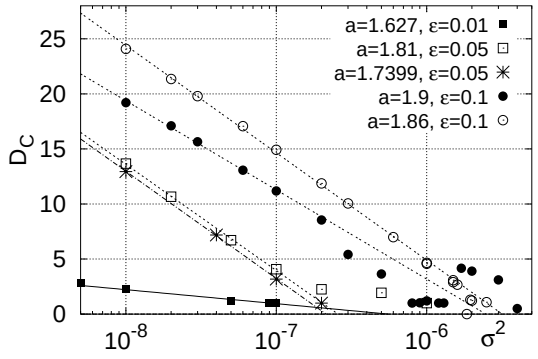

FIG. 3. The dimension of the collective motion is plotted as a function of noise intensity, obtained by the Lyapunov dimension. The Lyapunov exponents are calculated as growth rates of the tangent vector around the orbit, obtained by Eq. (4). For numerical calculation, $\rho_n(x)$ is approximated by a sufficiently large dimensional vector, and the tangent vectors are orthonormalized at each time step.

for an idealized example. In most cases, however, it is quite difficult and may well be impossible to obtain such a description. Thus, we examine the linear response of the system against infinitesimal perturbation on the macroscopic variables and obtain the variational equation describing the evolution of the small deviation of the macroscopic variables in a neighborhood of a trajectory.

In the present case, since the elements interact only through the mean-field value, it is quite natural to expect that the behavior of the mean-field value can be consistently described by itself. We expect that the effective number of the dimension of the collective motion gives substantial agreement with the Lyapunov dimension of the macroscopic dynamics estimated in the above mentioned way. We concentrate on the small σ regime and give a qualitative explanation for the relation (5).

If we consider small deviations η_n of h_n, then η_n is regarded as a function of $\{\eta_{n-1}, \eta_{n-2}, \ldots\}$. The evolution of η_n from the unperturbed orbit h_n is given by

$$\eta_n = \sum_{\tau=1}^{\infty} L_\tau \eta_{n-\tau} + O(\eta^2), \tag{7}$$

where L_τ is a coefficient to give the linear response of the mean-field value at the n step to the displacement at the $n - \tau$ step. The number of the Lyapunov dimension of the mean-field dynamics is estimated from the eigenvalues of this linear regression [17].

First we estimate L_τ from the dynamics of the distribution function given by Eq. (4). In the small noise limit ($\sigma \to 0$), from Eqs. (3) and (4), $h_n = \int F_n^{(\tau+1)}(x)\rho_{n-\tau}(x)\,dx$, where $F_n^{(\tau)}(x) \equiv F_n \circ F_{n-1} \circ \cdots \circ F_{n-\tau+1}(x)$. Considering an infinitesimal displacement applied at the $n - \tau$ step, L_τ is given by

$$L_\tau = \epsilon \int dx \, \frac{dF_n^{(\tau)}(x)}{dx} \, \rho_{n-\tau+1}(x). \tag{8}$$

For $\tau \gg 1$, $dF_n^{(\tau)}(x)/dx$ in Eq. (8) changes its sign quite frequently in x. Now let us consider the partition of x at the points such that $F_n^{(\tau)}(x) = 0$. Denoting the typical value of $|dF_n^{(\tau)}(x)/dx|$ by $d(\tau)$, the interval of partitions is estimated at $1/d(\tau)$, which decreases rapidly with τ. Since the integration in a partition becomes zero if $\rho(x)$ stays constant in that partition, the partitions where $\rho_n(x)$ changes drastically in x contribute to the estimation of L_τ much more than the partitions where $\rho_n(x)$ does not change so much.

In the case of a small noise limit ($\sigma \to 0$), the most drastic change of $\rho_n(x)$ comes from the inverse square-root singularities, which is the characteristic structure of the distribution function for the logistic map. Hence, the integration in the partitions containing the characteristic structure in $\rho_n(x)$ is estimated at $O[\sqrt{d(\tau)}]$ [18]. $d(\tau)$ is roughly estimated at $e^{\lambda_m \tau}$ for $\tau \gg 1$, where λ_m is the Lyapunov exponent of the local mapping [19]. Consequently, the response L_τ to the perturbation grows exponentially with the rate $\frac{1}{2}\lambda_m$.

Even in the presence of finite amplitude of the noise, the above order estimation for L_τ is still valid for τ smaller than $\tau_c \equiv -\log\sigma/\lambda_m$, where the typical width of the partitions becomes comparable with the typical amplitude of the noise, i.e., $1/d(\tau) \sim e^{-\lambda_m \tau} = \sigma$.

For larger $\tau > \tau_c$, however, the effect of noise in smoothening the distribution $\rho(x)$ appears so that L_τ will start to decay with τ.

Partially integrating (8), we obtain

$$L_\tau = -\epsilon \int dx \, \tilde{F}_n^{(\tau)}(x) \, \frac{d\rho_{n-\tau+1}(x)}{dx}, \tag{9}$$

with $\tilde{F}_n^{(\tau)}(x) = F_n^{(\tau)}(x) - \overline{F}_n^{(\tau)}$, where $\overline{F}_n^{(\tau)}$ is the average value of $F_n^{(\tau)}(x)$ over the support of $\rho_n(x)$. With the increase of τ, $\tilde{F}_n^{(\tau)}(x)$ becomes a rapidly oscillating function about zero mean in x, and the integration of $\tilde{F}_n^{(\tau)}(x)$ over any finite range within the support of $\rho_n(x)$ will approach zero [20]. Since $|\frac{d\rho_{n-\tau+1}(x)}{dx}|$ is uniformly bounded due to the existence of the noise, L_τ converges to zero. Hence, as far as the linear stability is concerned, the mean-field value is not sensitive to the perturbation on the mean-field values of sufficiently long time steps [$\gtrsim O(\tau_c)$ steps] ago. Thus we can consider a dynamics of h_n as a function of the mean-field values of the past $O(\tau_c)$ steps as Eq. (6) at least in the neighborhood of the orbit.

In summary, the amplitude of L_τ grows with τ as $L_\tau \sim e^{(1/2)\lambda_m \tau}$ for $\tau < O(\tau_c)$, whereas it starts to decay for $\tau > O(\tau_c)$. Thus for sufficiently large τ_c, i.e., for sufficiently small σ, the number of positive eigenvalues around zero for Eq. (7) is estimated at $O(\tau_c)$. Since we have to consider the contribution only from the latest $O(\tau_c)$ steps, the dimension of this dynamical system can be at most $O(\tau_c)$. Hence, the dimension of the mean-field dynamics is within the order of τ_c. Accordingly the number of effective degrees of freedom of the mean-field dynamics grows as $-\log\sigma$ with the decrease of σ, and can grow arbitrary large as σ approaches zero.

In the present Letter, we have shown that the noise in a microscopic level reduces the complexity of the collective motion, which is characterized by the number of degrees of freedom. It is shown numerically that the dimension of the dynamics of the Perron-Frobenius equation (4) satisfies (5). On the other hand, analysis on the mean-field dynamics also supports (5). Hence, the number of effective degrees of freedom of the collective motion in the present GCM is concluded to satisfy the relation (5).

Such a relation is expected to hold when the collective motion keeps high-dimensional motion with microscopic chaos. High-dimensional collective motions are supported by the exponential growth of the linear response coefficient. When the distribution has peak concentrations within a small width δx, such an exponential growth is expected until $\tau \sim \log\delta x$ and thus a high-dimensional collective motion appears [21]. The high-dimensional collective motion is also expected in a system that consists of elements with distributed parameters, whereas such "heterogeneity" may reduce the number of the dimension of the collective motion [9,13]. Even when the local dynamics is given as a higher-dimensional nonhyperbolic map or a chaotic flow, the present argument on the collective motion is expected to hold. Hence, the logarithmic dependence of the dimension upon the noise intensity given by (5) will be observed in a broad range of systems.

The induced regularity by the addition of noise was also reported as noise-induced order (NIO) in a one-dimensional map [3]. The mechanism to induce regularity in our system has a similarity with the NIO case, in the point that the external noise destroys a dynamical structure which causes the irregular behavior. In NIO, the noise reduces the concentration of measure in the instability region where the intermittent behavior is generated. On the other hand, in the present case, the noise smooths the singularity of the distribution function, that is, the source of high-dimensional instabilities of the collective motion.

In the case of NIO, however, the motion is not completely regular, since the noise is imposed upon the observed variable itself. On the other hand, in the present case, the noise is imposed on the microscopic variables, whereas the macroscopic collective variable is observed. Thus, the noise-induced motion is *deterministic* and low dimensional in the thermodynamic limit.

We should also mention that our present result may be applicable to experimental systems, such as fluid turbulence and neural systems consisting of a huge number of neurons with nonlinear dynamics. By controlling the thermal noise or some external fluctuations, we expect to observe the gradual reduction of the dimension and noise-induced low-dimensional motions, and hence the existence of the collective chaos will be clarified.

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[15] In order to bound the system within a finite region, "cutoff" of ξ is introduced, so that $f(f(1 + \xi) + \xi) > f(1 + \xi)$. Since we study the small σ regime in this Letter, the cutoff effect is negligible in our numerical results.
[16] Since we study a discrete time system, the quasiperiodicity means that the frequency of the mean-field motion is irrational.
[17] This does not hold if the effects of perturbations at different time steps could cancel out completely, as in the case that $\rho_n(x)$ contracts into a δ function. In the present case, however, as long as $\rho_n(x)$ has a continuous support, as is expected from the microscopic high-dimensional chaos, the effect of η_n cannot disappear completely by the perturbation at the other time step.
[18] $\int_c^{c+1/d(\tau)} d(\tau) \cdot |x - c|^{-1/2} dx \sim \sqrt{d(\tau)}$ around the singular structures.
[19] Here we neglect the fluctuation of the expansion rate.
[20] Here we assume the map $F_n^{(\tau)}(x)$ has mixing property, which seems natural when $\rho_n(x)$ has a continuous support.
[21] Such a structure is generic in the GCM with a logistic-type map $f(x) = 1 - a|x|^\alpha$ $(\alpha \geq 1)$, where the high-dimensional collective motions are observed [10].